

# Detection of the Peak of an Arbitrary Spectrum

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**Summary**—A new procedure is described for determining that frequency at which the spectrum of a signal has its absolute peak. The salient feature of the procedure is that it does *not* explicitly involve the estimation of the spectrum of the signal itself. Specifically, it is shown that the limit of the iterated normalized autocorrelation [see (8) and (9)] of a function,  $f(t)$ , is a pure cosine wave whose frequency corresponds to the location of the peak of the spectrum of  $f(t)$ .

Furthermore, if one is willing to accept an estimated peak frequency of maximum energy to within a given finite spectral resolution, then the procedure terminates after a specified finite number of iterations. Results from a computer simulation of the procedure are described. The areas of application of this procedure are discussed, and the results indicate that this method of detecting a signal (*i.e.*, by the peak of its spectrum) merits further consideration. It is important to note that the consideration of random processes has not been undertaken in this initial study; the results apply to the spectral peak of a deterministic signal only.

## I. INTRODUCTION

IN COMMUNICATIONS engineering, it is often useful to be able to extract, from an incoming signal, that frequency which contains a greater (power or energy) density<sup>1</sup> than any other frequency. In this study, a new procedure is described for determining that frequency; the significant aspect of the procedure is that the spectrum of the signal need not be calculated.

The fundamental theorem of Section III describes the mathematically interesting result in the case where we carry our procedure to the limit. The more useful theorem in Section V, however, describes a realizable procedure for determining the frequency of maximum power or energy to within a finite spectral resolution. We begin by defining the quantities basic to the procedure.

## II. DEFINITIONS

Consider that class of real functions or signals  $f(t)$  whose autocorrelation function,  $g(t)$  has the following properties,

$$0 < g(0) < \infty \quad (1)$$

$$\int_{-\infty}^{\infty} |g(t)| dt < \infty \quad (2)$$

$$g(t) \text{ is continuous in } t. \quad (3)$$

Since  $g(t)$  is an autocorrelation, it is therefore an even function of its argument. In Section IV, we modify these

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<sup>1</sup> Whether this is the energy density spectrum or power density spectrum depends upon the class that the signal falls in. See Section II.

restrictions to allow periodic and discrete time functions. For the above class, we define functions  $f(t)$  to be of *finite energy* if

$$0 < \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty \quad (4)$$

in which case

$$g(t) = \int_{-\infty}^{\infty} f(\tau)f(t + \tau) d\tau. \quad (5)$$

Further, we define functions  $f(t)$  to be of *finite average power* if

$$0 < \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt < \infty \quad (6)$$

in which case

$$g(t) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)f(t + \tau) d\tau. \quad (7)$$

Functions of finite energy or of finite average power therefore satisfy (2), and if their autocorrelation function satisfies (1) and (3), then our results hold.<sup>2</sup>

We now define a set of normalized autocorrelation functions,  $R_n(t)$  as

$$R_0(t) = \frac{g(t)}{g(0)} \quad (8)$$

$$R_n(t) = \frac{\int_{-\infty}^{\infty} R_{n-1}(\tau)R_{n-1}(t + \tau) d\tau}{\int_{-\infty}^{\infty} R_{n-1}^2(\tau) d\tau} \quad n = 1, 2, 3, \dots \quad (9)$$

We recognize that  $R_1(t)$  is the normalized autocorrelation function of the normalized autocorrelation function  $R_0(t)$ , etc. We may thus consider  $R_n(t)$  to be the  $n$ th iterated normalized autocorrelation function of  $f(t)$ .

Consider the Fourier transform,  $S_n(\omega)$ , defined by

$$S_n(\omega) = \int_{-\infty}^{\infty} R_n(t)e^{-i\omega t} dt \quad \text{for all } n \quad (10)$$

and its inverse

$$R_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_n(\omega)e^{i\omega t} d\omega. \quad (11)$$

Defining as usual,  $S_n^2(\omega)$  to be the energy density spectrum of  $R_n(t)$ , we note that  $S_n(\omega)$  is proportional to the energy density spectrum of  $R_{n-1}(t)$  for  $n \geq 1$ ; in particular,  $S_0(\omega)$  is proportional to the spectral density<sup>3</sup> of  $f(t)$ . We

<sup>2</sup> The properties expressed in (1) and (3) are most easily stated in terms of  $g(t)$  and will be left in that form.

<sup>3</sup> That is, for signals of finite energy,  $S_0(\omega)$  represents their normalized *energy* density spectrum, whereas for signals of finite average power,  $S_0(\omega)$  represents their normalized *power* density spectrum.

also observe that  $R_n(t)$  and  $S_n(\omega)$  are both real even functions of their arguments.

### III. THE FUNDAMENTAL THEOREM

In the introduction, we stated that our procedure, consisting of operations limited strictly to the time domain, was able to determine the peak of the spectrum of a function  $f(t)$ . More specifically, we state

*Theorem 1:* If  $S_0(\omega)$  has a unique absolute maximum at  $\omega = \theta$ , viz.,

$$S_0(\theta) > S_0(\omega) \quad \text{for } \omega \neq \pm\theta \quad (12)$$

then

$$\lim_{n \rightarrow \infty} R_n(t) = \cos \theta t \quad (13)$$

and

$$\frac{S_n(\omega)}{S_n(\theta)} = \left( \frac{S_0(\omega)}{S_0(\theta)} \right)^{2^n} \quad (14)$$

*Corollary:* If  $S_0(\omega)$  has a finite number of equal absolute maxima at the frequencies  $\theta_k$  ( $k = 1, 2, \dots, K$ ), then

$$\lim_{n \rightarrow \infty} R_n(t) = \frac{1}{K} \sum_{k=1}^K \cos \theta_k t$$

and

$$\frac{S_n(\omega)}{S_n(\theta_k)} = \left( \frac{S_0(\omega)}{S_0(\theta_k)} \right)^{2^n}$$

The details of proof for this theorem and its corollary may be found in Kleinrock.<sup>4</sup> A short heuristic proof may be given as follows. Using the fact that the Fourier transform of the autocorrelation of a real even function is equal to the square of the Fourier transform of the function itself, we transform both sides of (9) to obtain

$$S_n(\omega) = \frac{S_{n-1}^2(\omega)}{\int_{-\infty}^{\infty} R_{n-1}(\tau) d\tau} \quad n = 1, 2, 3, \dots$$

Alternatively, we may use Parseval's theorem to rewrite this as

$$S_n(\omega) = 2\pi \frac{S_{n-1}^2(\omega)}{\int_{-\infty}^{\infty} S_{n-1}^2(\sigma) d\sigma} \quad n = 1, 2, 3, \dots$$

Forming the ratio  $S_n(\omega)/S_n(\theta)$ , we obtain

$$\frac{S_n(\omega)}{S_n(\theta)} = \left( \frac{S_{n-1}(\omega)}{S_{n-1}(\theta)} \right)^2 = \left( \frac{S_0(\omega)}{S_0(\theta)} \right)^{2^n}$$

Now since  $S_0(\omega)/S_0(\theta) < 1$  for all  $\omega \neq \pm\theta$ , we find that

$$\lim_{n \rightarrow \infty} \frac{S_n(\omega)}{S_n(\theta)} = \begin{cases} 0 & \omega \neq \pm\theta \\ 1 & \omega = \pm\theta. \end{cases}$$

From the definition of  $R_n(t)$ , we observe that  $R_n(0) = 1$  and thus, the total area under  $S_n(\omega)$  is constant and equal to  $2\pi$  for all  $n$ . Consequently, since  $S_n(\omega)$  is vanishingly small compared to  $S_n(\theta)$  for all  $\omega \neq \pm\theta$ , we recognize that as  $n \rightarrow \infty$ ,  $S_n(\omega)$  must approach two impulses centered at  $\omega = \pm\theta$ . Clearly, the transform of such a function is a pure cosine wave. Summarizing, then, the theorem is based upon the fact that raising a function to an arbitrarily large exponent and then normalizing this exponentiated function to have constant area, magnifies the value of the original function at its peak and shrinks the value of the function elsewhere.

### IV. EXTENSION TO PERIODIC AND DISCRETE TIME FUNCTIONS

In order to extend Theorem 1 (and its corollary) to periodic and/or discrete time functions, we need merely replace certain of the integrals of Section III with the expressions described below.

For *continuous periodic* functions, our results hold if we redefine the limits of all previous time integrals to extend over a single period,  $T_0$  (say), and if we redefine all integrations with respect to  $\omega$  as sums over the discrete set of harmonic frequencies, viz.,

$$\int_{-\infty}^{\infty} x(t) dt \rightarrow \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) dt \quad (15)$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} y(\omega) d\omega \rightarrow \sum_{m=-\infty}^{\infty} y(\omega_m) \quad (16)$$

where

$$\omega_m = \frac{2\pi m}{T_0}$$

For these periodic functions, we obtain periodic autocorrelation functions  $g(t)$  where

$$g(t) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(\tau) f(t + \tau) d\tau \quad (17)$$

In addition,<sup>5</sup> from (9), (11), and (16),

$$\sum_{m=-\infty}^{\infty} S_n(\omega_m) = 1 \quad (18)$$

For *discrete aperiodic* time functions, our results hold if we replace all time integrals with infinite summations over the discrete time variable, and also redefine the limits on all integrations over  $\omega$  to extend over the finite range  $-\pi/\Delta t \leq \omega \leq \pi/\Delta t$  where  $\Delta t$  is the uniform increment between adjacent time samples. That is,

$$\int_{-\infty}^{\infty} x(t) dt \rightarrow \sum_{m=-\infty}^{\infty} x(t_m) \quad (19)$$

<sup>5</sup> Note that for continuous aperiodic time functions,

$$\int_{-\infty}^{\infty} S_n(\omega) d\omega = 2\pi R_n(0) = 2\pi.$$

<sup>4</sup> L. Kleinrock, "Detection of the Energy Peak of an Arbitrary Signal," M. I. T. Lincoln Laboratory, Lexington, Mass. Tech. Rept. (TR-325).

and

$$\int_{-\infty}^{\infty} y(\omega) d\omega \rightarrow \Delta t \int_{-\pi/\Delta t}^{\pi/\Delta t} y(\omega) d\omega \quad (20)$$

where

$$t_m = m\Delta t. \quad (21)$$

For these discrete functions, we obtain discrete autocorrelation functions  $g(t_m)$ . Furthermore, we find that

$$\int_{-\pi/\Delta t}^{\pi/\Delta t} S_n(\omega) d\omega = \frac{2\pi}{\Delta t}. \quad (22)$$

For *discrete* functions (of increment  $\Delta t$ ) which are *periodic* with period  $P\Delta t$ , we must make the following changes,<sup>6</sup>

$$\int_{-\infty}^{\infty} x(t) dt \rightarrow \sum_{i=0}^{P-1} x(t_i) \quad (23)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} y(\omega) d\omega \rightarrow \frac{1}{P} \sum_{m=-P/2}^{P/2} y(\omega_m) \quad (24)$$

where

$$t_i = i\Delta t$$

$$\omega_m = \frac{2\pi m}{P\Delta t}$$

For these discrete periodic functions, we obtain discrete periodic autocorrelation functions  $g(t_i)$ . Moreover, we have

$$\sum_{m=-P/2}^{P/2} S(\omega_m) = P. \quad (25)$$

## V. FINITE SPECTRAL RESOLUTION— A FEASIBLE TIME DOMAIN PROCEDURE

The interesting feature of our iterated autocorrelation function is that one may determine the spectrum peak (at  $\omega = \theta$ , say) of a signal with operations strictly in the time domain. Generally, in practice, one has a signal  $f(t)$  of finite duration ( $T$ ) to consider.<sup>7</sup> As a result,  $R_0(t)$ , the autocorrelation of the signal will be zero outside the interval  $|t| \geq T$ . However, the formal procedure for calculating  $R_n(t)$  as described by (9) indicates that  $R_n(t)$  will, in general, be nonzero in the interval  $|t| \leq 2^n T$ . From a practical point of view, this requires an exponentially increasing complexity (in either equipment or computation). At the same time we have an exponential rate of convergence to our limit function where the *exponent* is  $2^n$  (as may be seen from (14)). It is clear that this rate of convergence, although exponential, is nevertheless dependent upon the shape of  $S_0(\omega)$ . Note, however, that the resolution is (theoretically) perfect; *i.e.*, we are guaranteed to converge on the value  $\theta$ , exactly.

The price for perfect resolution is, as always, extremely high and one that we are not willing (or able) to pay;

<sup>6</sup> For convenience, we assume  $P$  to be an even integer. When  $P$  is odd, we must alter the limits in (24) and (25) slightly.

<sup>7</sup> Therefore, the signal will be of finite energy (see (4)).

namely, we require an unbounded number of calculations if we insist on passing to the limit  $n \rightarrow \infty$ . In particular, two separate aspects of the complexity of the process grow without bound: the number of iterations,  $n$ ; and the range  $|t| \leq 2^n T$  in which  $R_n(t)$  must be calculated. If we are willing to sacrifice some resolution, then we may control both of these quantities as follows.

Let  $f(t)$  be the signal whose spectrum peak (at  $\omega = \theta$ ) we desire, where we assume  $f(t)$  to be of finite duration  $T$  seconds; *i.e.*,  $f(t) \equiv 0$  for  $t < 0$  and  $t > T$ . For this function, we make the additional assumptions stated in (1)–(3) of Section II. Define  $R_n(t)$  and  $S_n(\omega)$  as in (8)–(11). We introduce the *periodic function*  $f_p(t)$  of period  $T_0 \geq 2T$  such that

$$f_p(t + kT_0) = f(t) \quad 0 \leq t \leq T_0 \quad (26)$$

$$k = \dots, -2, -1, 0, 1, 2, \dots$$

For this function, we define  $S_n^{(P)}(\omega_m)$  and  $R_n^{(P)}(t)$  as in (4)–(7) with the changes indicated in (15)–(18); the superscript  $P$  serves to distinguish these from  $S_n(\omega)$  and  $R_n(t)$  which refer to  $f(t)$ .

In Theorem 2, we show that an estimate  $\hat{\theta}$  of the spectrum peak in  $f_p(t)$  is a close approximation to  $\theta$  when  $T_0$  is sufficiently large. In addition, we describe a procedure for obtaining this estimate to within a given accuracy after a finite number ( $n_0$ ) of iterations. Furthermore, since periodic functions have periodic autocorrelation functions, we recognize that  $R_n^{(P)}(t)$  is periodic (with period  $T_0$ ) for all  $n$ . Consequently, we need only carry out our calculations of  $R_n^{(P)}(t)$  over *one* period only. Thus, by agreeing to estimate  $\theta$  within a slight (arbitrarily small) error, we have been able to reduce our system to one which goes through a *finite number of iterations* over a *finite interval*  $(0, T_0)$  which is fixed with respect to the number of iterations. All this, of course, comes about by allowing an uncertainty of size  $\rho$  (say) in the estimate of the frequency of maximum spectral density.

The relationship between the time limited signal  $f(t)$ , and its periodic counterpart  $f_p(t)$ , needs further elaboration. Indeed, we recognize that the autocorrelation functions of these two signals obey an inverse of the sampling theorem. Specifically, since  $T_0 \geq 2T$ ,  $R_0^{(P)}(t)$  contains all the information about  $R_0(t)$ ; consequently,  $S_0^{(P)}(\omega_m)$  must be proportional to  $S_0(\omega)$  at  $\omega = \omega_m = 2\pi m/T_0$ . We may calculate this factor of proportionality from (10) and (15), *viz.*,

$$\frac{S_0^{(P)}(\omega_m)}{S_0(\omega_m)} = \frac{\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} R_0^{(P)}(t) e^{-i\omega_m t} dt}{\int_{-\infty}^{\infty} R_0(t) e^{-i\omega_m t} dt}$$

The integrand in the numerator disappears in the interval  $T < |t| < T_0/2$  and the integrand in the denominator disappears over the interval  $|t| > T$ . Therefore, since  $T_0/2 \geq T$ , we find that

$$S_0^{(P)}(\omega_m) = \frac{1}{T_0} S_0(\omega_m). \quad (27)$$

Recognizing that these two spectra are proportional is essential to one's understanding of Theorem 2. A typical case is shown in Fig. 1, where the continuous spectrum  $S_0(\omega)$  is shown as the continuous envelope, and the line spectrum  $T_0 S_0^{(P)}(\omega_m)$  is shown as vertical lines. Observe that the distance between "samples" of  $S_0(\omega)$  is  $2\pi/T_0$ . We define an integer  $M$  such that  $\omega_M$  and  $\omega_{M+1}$  surround the spectral peak at  $\omega = \theta$  (the same clearly holds true for  $-\omega_M$ ,  $-\theta$ , and  $-\omega_{M+1}$ ). Further, we define  $\mu$  to be that frequency at which  $S_0(\omega)$  has its second greatest local maximum.

In estimating  $\theta$ , two separate problems confront us: first, how do we eliminate competing peaks (such as at  $\omega = \pm\mu$ ); and secondly, how do we converge to the neighborhood of  $\theta$  within the absolute peak itself. The first problem may be solved by insisting that the spacing,  $2\pi/T_0$  be fine enough to insure that at least one sample in the neighborhood of  $\theta$  (either  $\omega_M$  or  $\omega_{M+1}$ ) is greater than the maximum possible sample near  $\omega = \mu$ . Thus, an important parameter of the spectrum  $S_0(\omega)$  may be expressed as a lower bound  $\Delta$  for the difference between these two peaks, *viz.*,

$$S_0(\theta) - S_0(\mu) \geq \Delta. \quad (28)$$

In considering the problem of convergence in the neighborhood of  $\theta$ , we recognize that the shape of the spectrum in this region is crucial. Specifically, we require that the spacing  $2\pi/T_0$  be small enough to guarantee that some lines in the discrete spectrum fall within the range of the dominant peak (the narrower the peak, the finer must be the spacing). On the other hand, too many spectral lines in the immediate vicinity of  $\theta$  will result in a large number of required iterations because the largest sample will not differ significantly from its neighbors. We may discuss the width or sharpness of the spectrum in the vicinity of its absolute peak by considering the curvature of  $S_0(\omega)$  in terms of the magnitude of its second derivative  $d^2 S_0(\omega)/d\omega^2$ . When the second derivative is large (in magnitude), then the peak is narrow, and vice versa. Thus, we are led to consider upper and lower bounds for this quantity in the neighborhood of  $\theta$ , *viz.*,

$$-A \leq \frac{d^2 S_0(\omega)}{d\omega^2} \leq -a \quad \text{for } |\omega - \theta| \leq \frac{4\pi}{T_0}. \quad (29)$$

The preceding discussion deals qualitatively with those factors which determine the required number of iterations and the spacing of samples in the discrete spectrum (which is governed by the period  $T_0$ ). In addition to these considerations, the sampling must be fine enough to satisfy the spectral resolution requirements of the user of these results. Theorem 2 may now be stated.

**Theorem 2:**<sup>3</sup> Consider any function  $f(t)$  of duration  $T$  seconds, with spectral peak at  $\omega = \pm\theta$ , whose energy density spectrum  $S_0(\omega)$  satisfies the conditions (28) and (29) with respect to the three positive parameters

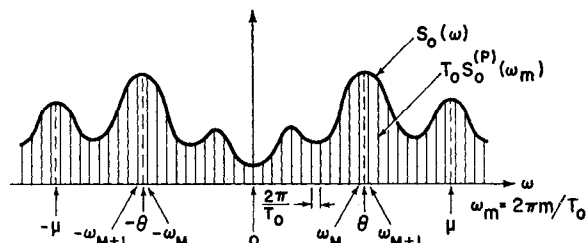


Fig. 1—Comparison between  $S_0(\omega)$  and  $S_0^{(P)}(\omega_m)$ .

( $a$ ,  $A$ ,  $\Delta$ ). Then, for a given required frequency resolution  $\rho$  there exists a procedure (defined below) which will calculate a number  $\hat{\theta}$  after  $n_0$  iterations of the autocorrelation of  $f_x(t)$  [see (26)] over the range  $|t| \leq T_0$  such that

$$|\hat{\theta} - \theta| \leq \rho$$

where

$$n_0 + 1 = \log_2 \frac{1}{\epsilon_0} \quad (30)$$

and

$$\epsilon_0 = \frac{2\pi^2}{T_0^3} a \quad (31)$$

and

$$T_0 = \max \left[ \pi \sqrt{\frac{A + 4a}{2\Delta}}, \frac{5\pi}{\rho}, 2T \right]. \quad (32)$$

Let us now describe the procedure by which we determine the estimate  $\hat{\theta}$  from  $R_n^{(P)}(t)$ . We recognize that if one line (say at  $\omega_m = \theta_p$ ) of the spectrum of the periodic function  $R_n^{(P)}(t)$  is "sufficiently" large<sup>9</sup> compared to all the other lines, then  $R_n^{(P)}(t)$  will appear as a "noisy" cosine wave at frequency  $\theta_p$  radians/sec. Our procedure then is to count the number of "noisy" zero-crossings of this function for a known interval of time; if we are successful in counting only the true zero-crossings of the pure cosine wave, then we can ascertain  $\theta_p$  exactly. Accordingly, we define a *noisy zero-crossing counter* as follows. First, we consider a  $\delta$ -threshold detector with hysteresis defined by the transfer characteristic shown in Fig. 2. Further define

$Z(t)$  = number of zeros (or counts) recorded by the noisy zero-crossing counter in  $t$  sec.

$$Z(0) = 1.$$

The noisy-crossing counter consists of a  $\delta$ -threshold detector with hysteresis followed by a simple counter which registers a count each time the detector changes state (in either direction) as in Fig. 3. It is shown in Kleinrock<sup>4</sup> that a setting of  $\delta = 1/9$  defines a noisy zero-crossing counter which produces an estimate  $\hat{\theta}$  consistent with Theorem 2. That is, for  $n \geq n_0$ , we get  $\delta_n \leq \delta = 1/9$  (where  $\delta_n$  is defined by (34) below).

<sup>9</sup> It is clear that by increasing  $T_0$ ,  $\theta_p$  can be made as close to  $\theta$  as one desires, and also, that by increasing the number of iterations,  $S_n^{(P)}(\theta_p)$  can be made arbitrarily large compared to all other spectral components (see Theorem 1).

<sup>3</sup> The rather lengthy proof of this theorem may be found in Kleinrock.<sup>4</sup>

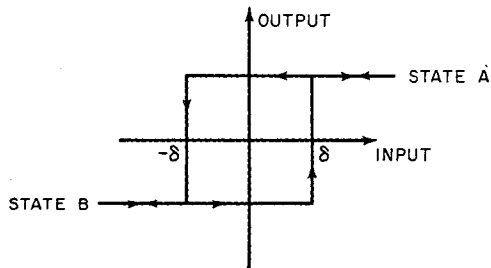


Fig. 2—Transfer characteristic of the  $\delta$ -threshold detector with hysteresis.

With no loss of generality, we may write

$$R_n^{(P)}(t) = B_n \cos \theta_p t + b_n(t). \quad (33)$$

This form exposes the pure cosine wave which we wish to detect, and groups the transform of all the other spectral components into the function  $b_n(t)$ . Define  $\delta_n$  by

$$|b_n(t)| < \delta_n \quad \text{for } |t| \leq T_0. \quad (34)$$

Then, if

$$B_n \geq 2\delta_n \quad (35)$$

we are guaranteed that  $Z(t)$  will increase by unity each time  $B_n \cos \theta_p t$  passes through zero, as may be seen from Fig. 4. That is, when  $B_n \cos \theta_p t$  is negative,  $R_n^{(P)}(t) < \delta_n$  and when  $B_n \cos \theta_p t$  is positive,  $R_n^{(P)}(t) > -\delta_n$  thus insuring that a single zero-crossing (which is equivalent to a sign change) of  $B_n \cos \theta_p t$  cannot generate more than one count. Furthermore, by insisting that  $B_n \geq 2\delta_n$ , we guarantee that each time  $B_n \cos \theta_p t$  passes through zero, we must get at least one count. Thus  $Z(t) - 1$  will count the number of zero-crossings of  $B_n \cos \theta_p t$  exactly. Now, since  $T_0$  is a multiple of the period of  $B_n \cos \theta_p t$ , we determine  $\theta_p$  from

$$\theta_p = \frac{Z(T_0) - 1}{T_0} \pi. \quad (36)$$

More generally, if we observe  $Z(t)$  for a time  $\tau$ , where  $0 \leq \tau \leq T_0$ , then we may ask how large an error is made in our estimation of  $\theta_p$ ; in particular, how large is  $|\theta_p - Z(\tau)\pi/\tau|$ . To answer this, we use Euclid's algorithm to express

$$\frac{\tau}{P} = q + \frac{r}{P}$$

where  $P$  is the period of  $\cos \theta_p t$  (i.e.,  $P = 2\pi/\theta_p$ ) and  $q$  and  $r$  are, respectively, the quotient and remainder of  $\tau/P$ , where, of course,  $r < P$ . Since we increment  $Z(t)$  twice for each period,  $P$ , of  $\cos \theta_p t$  we have  $q = [Z(\tau) - 1]/2$ . Thus,

$$\frac{\tau}{P} = \frac{Z(\tau) - 1}{2} + \frac{r}{P}$$

or, multiplying by  $2\pi/\tau$ , we obtain

$$\frac{2\pi}{P} = \frac{\pi Z(\tau)}{\tau} + \frac{\pi}{\tau} \left( 2 \frac{r}{P} - 1 \right).$$

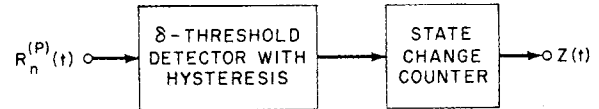


Fig. 3—The noisy zero-crossing counter.

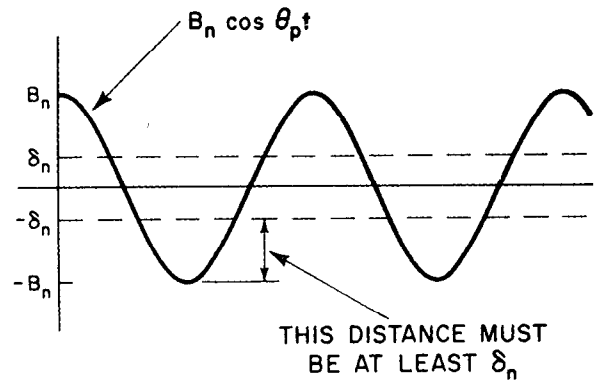


Fig. 4—Description of the relation between  $B_n$  and  $\delta_n$ .

Now, since  $\theta_p = 2\pi/P$ , and since  $r < P$ , we have

$$\left| \theta_p - \frac{Z(\tau)}{\tau} \pi \right| \leq \frac{\pi}{\tau}. \quad (37)$$

Thus, we conclude that the noisy zero-crossing counter determines from  $R_n^{(P)}(t)$ , the frequency  $\theta_p$  to within the accuracy described by (37). The error in this determination may be made arbitrarily small by increasing  $\tau$ . It turns out that a convenient value for  $\tau$  can be shown to be  $\tau = T_0/3$ .

## VI. EXPERIMENTAL RESULTS

The procedure described in Section V for determining the spectral peak of a finite duration signal to a finite resolution was simulated on a digital computer.<sup>10</sup> Results of the experimentation are shown in Fig. 5.

Specifically, the signal  $f(t)$  chosen was a small segment ( $T = 0.0256$  seconds) of human speech sampled at a 10 kc rate. Fig. 5 shows only one period (i.e., the interval  $0 \leq t \leq T_0$ ) of  $f_p(t)$  and  $R_n^{(P)}(t)$  for  $n = 0, 1, 2$ , as well as  $S_n(\omega)$  for  $n = 0, 1, 2, 3$ .  $T_0$  was chosen equal to  $2T$  for this experiment. Note that  $S_n(\omega)$  is shown only as a visual aid; its calculation was clearly *not* necessary for the generation of  $R_n^{(P)}(t)$ . We chose to show  $S_n(\omega)$  rather than  $S_n^{(P)}(\omega_m)$  for convenience of programming; as a result, we observe the  $([\sin a(\omega - \omega_m)]/a(\omega - \omega_m))^2$  envelope quite distinctly in  $S_n(\omega)$ .

One clearly observes the rapid convergence of both  $R_n^{(P)}(t)$  and  $S_n(\omega)$  to the frequency of maximum density.

<sup>10</sup> A program for simulating the iterated autocorrelation written by the author (and a spectrum analysis program written by C. Rader), was run on the M. I. T. Lincoln Laboratory TX-2 high speed digital computer.<sup>11</sup>

<sup>11</sup> J. M. Frankovich and H. P. Peterson, "A Functional Description of the Lincoln TX-2 Computer," *Proc. Western Joint Computer Conf.*, pp. 146-155; 1957.

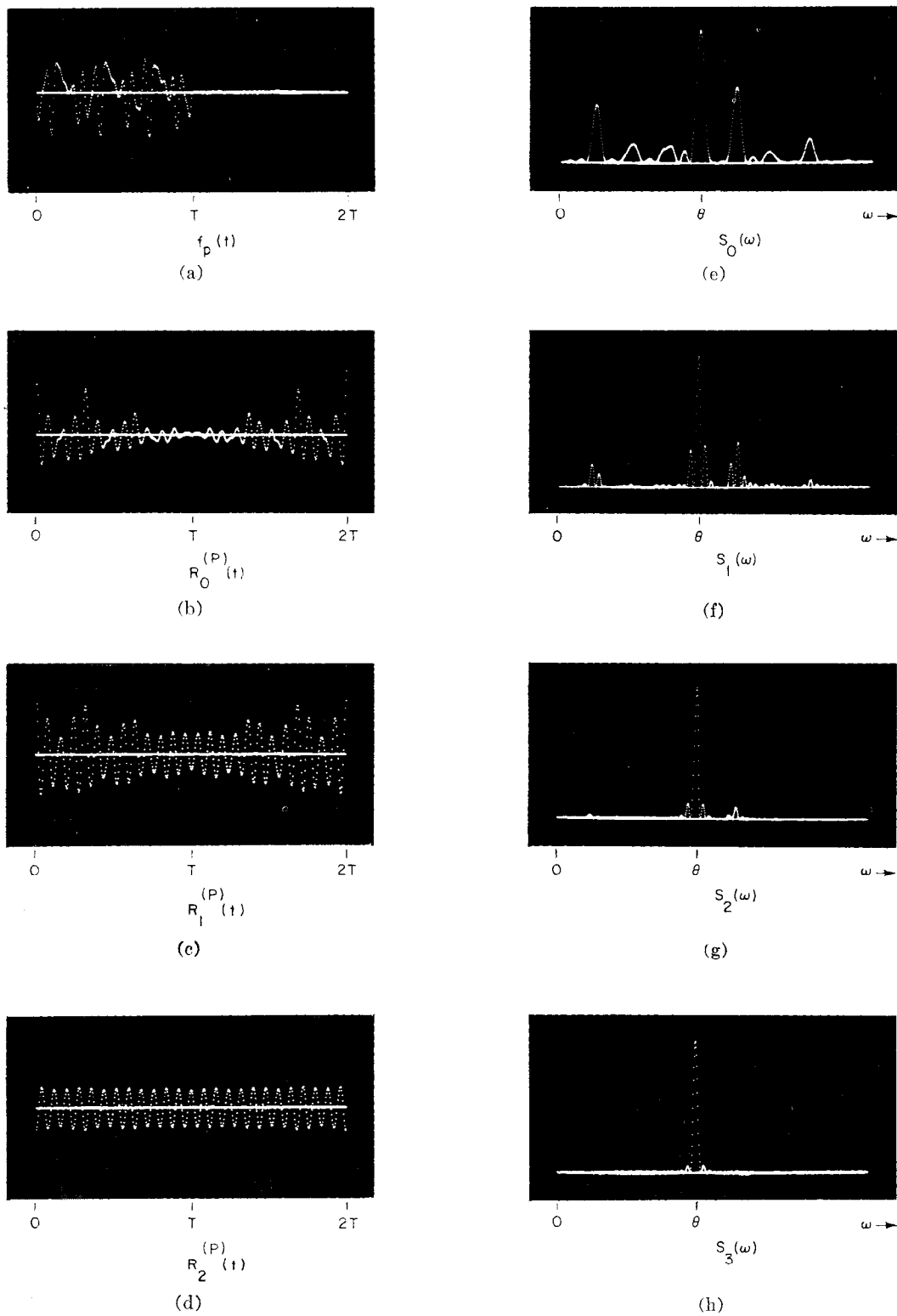


Fig. 5—The power spectrum  $S_n(\omega)$  and one period of the iterated autocorrelation function  $R_n^{(P)}(t)$  as a function of the number of iterations  $n$ .

## VII. APPLICATIONS AND CONCLUSIONS

Theorem 1 expresses the fundamental result that the limit of the iterated normalized autocorrelation function of a signal is a cosine wave at a frequency ( $\theta$ ) corresponding to the maximum peak of the signal's spectrum. However, two aspects of the procedure by which one arrives at this limit require unbounded complexity: first, the interval over which the  $n$ th iteration must be calculated (assuming the signal to be of finite duration  $T$ ) is  $|t| \leq 2^n T$ ; second, the number of iterations grows without bound. These two difficulties require unlimited equipment and time, respectively. Clearly, the reason behind these infinite operations is that we are asking for an absolutely perfect determination of  $\theta$ . Naturally, we are willing to accept some error in this determination in any practical situation. Taking advantage of this fact, we are able to establish Theorem 2 in which we offer a procedure for estimating  $\theta$  (to within an arbitrarily small, but finite, error) which requires a finite number of calculations over a fixed time interval. Thus, by accepting an error in the determination of  $\theta$ , we have been able to eliminate *both* undesirable aspects of the original procedure.

The procedure for obtaining the estimate ( $\hat{\theta}$ ) of  $\theta$  may be mechanized approximately as follows (see Fig. 6). The signal,  $f(t)$  would be stored on a tapped delay line (of  $T_0$  seconds). At time  $T_0$ , the impulse response of a linear filter<sup>12</sup> would be set equal to the values of the taps, and the output of the delay line would then feed into the filter. The output of the filter (now equal to the autocorrelation of the signal) would then be fed back into the delay line. The procedure is repeated  $n_0$  times after which, the output of the filter is sent through the noisy zero-crossing counter, which provides the estimate  $\hat{\theta}$ .

<sup>12</sup> This filter is  $2T_0$  seconds long, with its impulse response  $h(t)$  adjusted such that  $h(t + T_0) = h(t)$  thus representing the periodic version of the signal. In the figure, this linear filter is represented as a combination of a holding circuit and a second tapped delay line; the outputs are multiplied and summed to form the output of the filter.

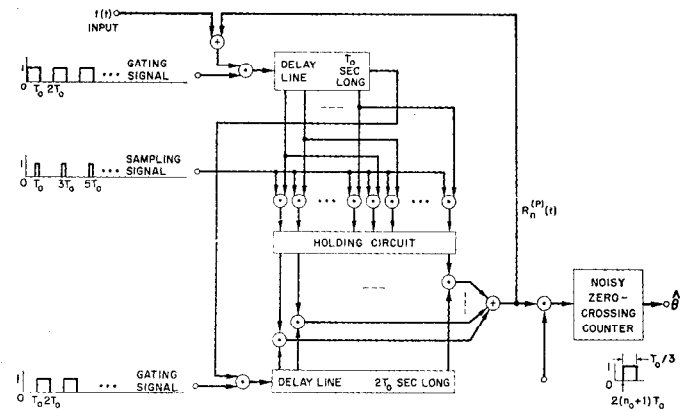


Fig. 6—Implementation of the procedure for detecting the frequency of maximum energy density.

The applications of these results appear to be numerous. The detection of the spectral peak of an arbitrary signal (for example the peak corresponding to a time series) is often of great interest, and may be detected as above. Furthermore, this method may be used for selecting that one out of  $N$  (say) possible signals transmitted over a communication link; in this application, it is necessary of course, that the location of the absolute peaks in the spectra of the  $N$  signals be distinct one from the other, such as is the case in frequency shift keying. Another application may be found in locating the peak frequency in the return signal from a Doppler radar. These are but a few of the possible applications of the procedure described.

The main conclusion to be drawn from this study is that the iterated autocorrelation procedure represents a new method for detecting the frequency of maximum spectral density of a signal. Some areas of application have been suggested above briefly, but more careful consideration and analysis must be undertaken before one can determine the advantages or disadvantages of this system compared to any other. In fact, one of the main purposes of presenting this material is to stimulate thinking about the possible applications and merits of this new detection approach.