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ELEC. ENG. DEPT.
By WMLS Date 7/24/61

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Information Flow in Large Communication Nets

Proposal for a Ph.D. Thesis

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May 31, 1961

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I. Statement of the Problem:

The purpose of this thesis is to investigate the problems associated with information flow in large communication nets. These problems appear to have wide application, and yet, little serious research has been conducted in this field. The nets under consideration consist of nodes, connected to each other by links. The nodes receive, sort, store, and transmit messages that enter and leave via the links. The links consist of one-way channels, with fixed capacities. Among the typical systems which fit this description are the Post Office System, telegraph systems, and satellite communication systems.

A number of interesting and important questions can be asked about this system, and it is the purpose of this research to investigate the answers to some of these questions. A partial list of such questions might be as follows:

(1) What is the probability density distribution for the total time lapse between the initiation and reception of a message between any two nodes? In particular, what is the expected value of this distribution?

(2) Can one discuss the effective channel capacity between any two nodes?

(3) Is it possible to predict the transient behavior and recovery time of the net under sudden changes in the traffic statistics?

(4) How large should the storage capacity be at each node?

(5) In what way does one arrive at a routing doctrine for incoming messages in different nets? In fact, can one state some bounds on the optimum performance of the net, independent of the routing doctrine (under some constraint on the set of allowable doctrines)?

(6) Under what conditions does the net jam up, i.e., present an excessive delay in transmitting messages through the net? The solution to this problem will dictate the extent to which the capacity of each link can be used (i.e., the ratio of rate to channel capacity, which is commonly known as the utilization factor).

(7) What are the effects of such things as additional intranode delays, and priority messages?

One other variable in the system is the amount of information that each node has about the state of the system (i.e., how long the queues are in each other node). It is clear that these are critical questions which need answers, and it is the intent of this research to answer some of them.

In attempting the solution of some of these problems, it may well be that the study of a specific system or application will expose the basis for an understanding of the problem. It is anticipated that such a study, as well as a simulation of the system on a digital computer, will be undertaken in the course of this research.

II. History of the Problem

The application of Probability Theory to problems of telephone traffic represents the earliest area of investigation related to the present communication network problem. The first work in this direction dates back to 1907 and 1908 when E. Johannsen [1][1,2] published two essays, the one dealing with delays to incoming calls in a manual telephone exchange, and the other being an investigation as to how often subscribers with one or more lines are reported "busy." It was Dr. Johannsen who encouraged A. K. Erlang to investigate problems of this nature. Erlang, an engineer with the Copenhagen telephone exchange, made a number of major contributions to the theory of telephone traffic, all of which are translated and reported in [1]; his first paper (on the Poissonian distribution of incoming calls) appeared in 1909 and the paper containing the results of his main work was published in 1917, in which

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[1] Numerals in square brackets refer to the bibliography.

[2] Reference to Johannsen's work will be found in [1], page 16.

he considered the effect of fluctuations of service demands on the utilization of the automatic equipment in the telephone exchange.

A few other workers made some contributions in this direction around this time, and a good account of the existing theories up to 1920 is given by O'Dell [2,3]; his principal work on grading appeared in 1927. Molina[4,5] was among the writers of that time, many of whom were concerned mainly with attempts at proving or disproving Erlang's formulas, as well as to modify these formulas.

The theory of stochastic processes was developed after Erlang's work. In fact it was Erlang who first introduced the concept of statistical equilibrium, and called attention to the study of distributions of holding times and of incoming calls. Much of modern queuing theory is devoted to the extension of these basic principles with the help of more recent mathematical tools.

In 1928, T.C. Fry [6] published his book (which has since become a classic text) in which he offered a fine survey of congestion problems. He was the first to unify all previous works up to that time. Another prominent writer of that period was C. Palm [7,8], who was the first to use generating functions, in studying the formulas of Erlang and O'Dell. His works appeared in 1937-1938. During this time, a large number of specific cases were investigated, using the theories already developed, in particular lost call problems. Both Fry and Palm formulated equations (now recognized as the Birth and Death equations) which provide the foundation for the modern theory of congestion.

In 1939, Feller [9] introduced the concept of the Birth and Death process, and ushered in the modern theory of congestion. His application was in physics and biology, but it was clear that the same process characterized many models useful in telephone traffic problems. Numerous applications of these equations were made by Palm [10] in 1943 In 1948, Jensen (see [1]) also used this process, without mentioning its name, for the elucidation of Erlang's work. Kosten [11], in 1949, studied the probability of loss by means of generalized Birth and Death equations. Waiting line and trunking problems were explained by Feller [12] in his widely used book on probability, making use of the theory of stochastic processes. At around 1939, the problems of waiting lines and trunking problems in telephone systems were taken up more by mathematicians than by telephone engineers.

In 1950, C.E. Shannon [13] considered the problem of storage requirements in telephone exchanges, and concluded that a bound can be placed on the size of such storage, by estimating the amount of information used in making the required connections. In 1951, F. Riordan [14] investigated a new method of approach suitable for general stochastic processes. R. Syski [15], in 1960, published a fine book in which he presented a summary of the theory of congestion and stochastic processes in telephone systems, and also cast some of the more advanced mathematical descriptions in common engineering terms.

In the early 1950's, it became obvious that many of the results obtained in the field of telephony were applicable in much more general situations, and so started investigations into waiting lines of many kinds, which has developed into modern Queuing Theory, itself a branch of Operations Research. A great deal of effort has been spent on single node facilities, i.e., a system in which "customers" enter, join a queue, eventually obtain "service" and upon completion of this service, leave the system. P.M. Morse [16] presents a fine introduction to such facilities in which he defines terms, indicates applications, and outlines some of the analytic aspects of the theory. P. Burke [17], in 1956, showed that for independent inter-arrival times (i.e., Poisson arrival), and exponential distribution of service times, the inter-departure times would also be independent (Poisson). In 1959, F. Foster [18] presented a duality principle in which he shows that reversing the roles of input (arrivals) and output (service completions) for a system will define a dual system very much like the original system. In contrast to the abundant supply of papers on single node facilities, relatively few works have been published on multi-node facilities (which is the area of interest to this thesis). Among those papers which have been presented is one by G.C. Hunt [19] in which he considers sequential arrays of waiting lines. He presents a table which gives the maximum utilization factor (ratio of average arrival rate to maximum service rate) for which steady state probabilities of queue length exist, under various allowabal queue lengths between various numbers of sequential service facilities. J.R. Jackson [20], in 1957, published a paper in which he investigated networks of waiting lines. His network consisted of a number of service facilities into which customers entered both from external sources as well as after having completed service in another facility. He proves a theorem which stated roughly, says that a steady state distribution for the system state exists, as long as the effective utilization factor for each facility is less than one, and in fact this distribution takes on a form identical to the solution for the single node case.[3] In 1960, R. Prosser [21] offered an approximate analysis of a random routing procedure in a communication net in which he shows that such procedures are highly inefficient but extremely stable (i.e., they degrade gracefully under partial failure of the network).

The two important characteristics of the communication nets that form the subject of this thesis are (1) the number of nodes in the system is large, and (2) each node is capable of storing messages while they wait for transmission channels to become available. As has been pointed out, Queuing Theory has directed most of its effort so far, toward single node facilities with storage. There has been, in addition, a considerable investigation into multi-node nets, with no storage capabilities, mainly under the title of Linear Programming (which is really a study of linear inequalities and convex sets). This latter research considers, in effect, steady state flow in large connected nets, and has yielded some interesting results. One problem which has attracted a lot of attention is the shortest route problem (also known as the travelling salesman problem). M. Pollack and W. Wiebenson [22] have presented a review of the many solutions to this problem, among which

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[3]Jackson's work is discussed in detail in Section IV.

are Dantzig's Simplex Method, Minty's labelling method, and the Moore-D'Esopo method. W. Jewell [23] has also considered this problem in some greater generality, and, by using the structure of the network and the principle of flow conservation, has extended an algorithm due to Ford and Fulkerson in order to solve a varied group of flow problems in an efficient manner. R. Chien [24] has given a systematic method for the realization of minimum capacity communication nets from their required terminal capacity requirements (again considering only nets with no storage capabilities); a different solution to the same problem has been obtained by Gomory and Hu [25]. In 1956, P. Elias, A. Feinstein, and C.E. Shannon [26] showed that the maximum rate of flow through a network, between any two terminals, is the minimum value among all simple cut-sets. Also, in 1956, Z. Prihar [27] presented an article in which he explored the topological properties of communication nets; for example, he showed matrix methods for finding the number of ways to travel between two nodes in a specific number of steps.

In 1959, P.A.P. Moran [28] wrote a monograph on the theory of storage. The book describes the basic probability problems that arise in the theory of storage, paying particular attention to problems of inventory, queuing, and dam storage. It represents one of the few works pertaining to a system of storage facilities.

The results from Information Theory [29] also have relation to the communication net problem considered here. Most of the work there has dealt with communication between two points, rather than communication within a network. In particular, one of the results says that there is a trade-off between message delay and probability of error in the transmitted message. Thus if delays are of no consequence, transmission with an arbitrarily low probability of error can be achieved. However, it is not obvious as yet, what effect such additional intranode delays would have in a large network of communication centers; it seems that some maximum additional delay exists, and if so, this would restrict the use of coding methods, and perhaps put a non-zero lower bound on the error probability.

III. Discussion of Proposed Procedure

The problems associated with a multi-terminal communication net, as posed in the first section, appear to be too difficult for analysis, in an exact mathematical form. That is to say, the calculation of the joint distribution of traffic flow through a large (or even small) network is extremely difficult. Even for networks in which no feedback is present, the mathematics is unmanageable; and for those with feedback, it seems hopeless to attempt an exact solution. The question, then, is to what degree, and in what fashion can we simplify this problem?

Since it is the complicated interconnections that cause most of the trouble, one would like to isolate each node, and perform an individual analysis on it, under some boundary constraints. The node could then be represented by the results of such an analysis. In particular, it is hoped that the node representation would be sufficiently complete, by the use of perhaps two numbers, these numbers being the mean and variance of the traffic handled by the node. Thus, instead of having to derive the complicated joint distribution of the traffic in the network, one may be able to make a fairly accurate characterization by specifying two (or at most a few) parameters.

This approach is not completely naive and without justification. Consider the linear programming techniques [21-26] mentioned in the second section of this proposal. The problems handled by such techniques have a great deal in common with the communication problem at hand. Their problem is that of solving networks in which the commodity (e.g., water, people, information) <u>flows steadily</u>. A typical problem would be that of finding the set of solutions (commonly referred to as feasible solutions) which would support a given traffic flow in a network. A solution would consist of specifying the flow capacity for each link between all pairs of nodes. In general, a large number of solutions exist, and a lot of effort has been spent in minimizing the total capacity used for such a problem. One obvious requirement is that the average traffic entering any node must be less than the total capacity leaving the node. Notice that the important statistic here is the <u>average</u> <u>traffic</u> <u>flow</u>, and if the flow is <u>steady</u>, then we have a deterministic problem. Now, in what way does this problem differ from the problem considered in this proposal? Clearly, the difference is that we do not have a steady flow of traffic. Rather, our traffic comes in spurts, according to some probability distribution. Consequently, we must be prepared to waste some of our channel capacity, i.e., the channel will sometimes be idle. A good measure of how non-steady our traffic is, is the variance of the traffic. That is, for zero variance, we are reduced to the special case above, namely, steady flow. As the variance goes up, we can say less and less about the arrival time, and the traffic becomes considerably more random in time. Thus, it is reasonable to expect that the two important parameters which characterize our traffic are the mean and variance of the flow. Notice that a necessary, but clearly not sufficient condition for a feasible solution to our problem is that the average traffic entering the node must be less than the total capacity of channels leaving the node. In 1951, Kendall [30] showed for a single node with Poisson inter-arrival times (at a rate λ per sec), an infinite allowable queue, and an arbitrary service distribution (with mean $1/\mu$ and variance v), that the expected waiting time in the system, E(t) was

$$E(t) = (1/\mu) + \frac{(1/\mu)^2 + v}{2[(1/\lambda) - (1/\mu)]}$$

This clearly shows a linear dependence on the variance.

Reference has already been made to J.R. Jackson [20]. The assumptions that he made in his analysis of networks of waiting lines was that the arriving traffic at each node had a Poisson distribution, that the service time was exponentially distributed, and that infinite queues at each node were allowed. With these assumptions, he was able to derive the distribution of traffic at all the nodes. It is important to analyze his assumptions carefully. The Poisson assumption effectively characterized the traffic with two parameters. This same assumption also effectively decoupled the nodes from each other. His results state that if the mean traffic satisfies the necessary condition stated in the previous paragraph, then the resultant traffic can be characterized by a two parameter description.

The question of queuing discipline is an interesting one, and one which causes some difficulties. That is, the node must decide on a method for choosing some member in the queue to be served next. An interesting simplication tothis question, and perhaps a key to the solution of the network problem may be obtained as follows. Consider that class of queue disciplines which require that a channel facility never remain idle, as long as the queue is non-empty (clearly this is a reasonable constraint). Now, adopting a macroscopic viewpoint, (i.e., removing all labels from the members in the queue), what can be said about the mean and variance of the waiting time distribution for this class of queue disciplines? It seems that some reasonably tight bounds might exist for this distribution, independent of the particular discipline used. Perhaps some other restriction on the class of disciplines will be required in order to obtain meaningful results. However, under such a set of restrictions, if we can characterize the queue sufficiently well, we may then be placed in a position to obtain some overall behavior for the network. All of the queuing problems solved to date, have considered a particular queue discipline (the microscopic viewpoint), and so the results have been specialized to an extremely large degree. Adopting the macroscopic viewpoint seems to be a natural step, and it is the intention of this research to investigate this avenue.

There appear to be a number of conflicting interests in a network of this type. The things to be considered are: storage capacity at each node; channel capacity at each node; and message delay. There exists a trading relationship among these quantities, and it is necessary to attach some quantitative measures to this trade. In fact, if one wishes to generalize one step further, one can consider a multi-terminal communication system, in which the signals are perturbed by noise in the system. Information theory tells us how to combat this disturbance, and the solution introduces additional delays in message transmission and reception. What effect these additional delays will have on the system is not clear; in fact it becomes difficult to state just what the overall capacity is for such a situation. Questions such as these are extremely important, and deserve attention.

From the statements in this section, it is clear that an approximate analysis is all that can easily be obtained for the network under consideration. Hopefully, the approximate answers will be reasonably useful. One way to check the utilization of the results is simulation. It is fully expected that, in the course of this research, a simulated net of this type will be programmed on one of the local digital computers. The author has access to the Lincoln Laboratory TX-2 computer, as well as the IBM 709. This simulation study should serve as a useful check on the results and perhaps, will also serve as a guide into the research.

IV. Preliminary Investigation

In this section, certain results will be presented, which have been obtained in the preliminary investigation already undertaken. The proofs of the new theorems will be left for the Appendix.

The point of departure is a theorem due to Jackson [20] which has already been referred to. He considers a situation in which there are M departments, the m^{th} department having the following properties ($m = 1, 2, \ldots, M$):

1. N_m servers
2. Customers from outside the system arrive in a Poisson-type time series at mean rate λ_m (additional customers will arrive from other departments in the system).
3. Service is on a first come, first served basis, with an infinite storage available for overflow; the servicing time being exponentially distributed with mean $1/\mu_m$.
4. Once served, a customer goes immediately from department m to department k with probability θ_{km}; his total service is completed (and he then leaves the system) with probability

$$1 - \sum_k \theta_{km} .$$

Property 4 is the basis on which Jackson calls this system a network of waiting lines. Defining Γ_m as the average arrival rate of customers at department m from all sources, inside and outside the system, Jackson states that

$$\Gamma_m = \lambda_m + \sum_k \theta_{mk} \Gamma_k \tag{1}$$

Now, defining n_m as the number of customers waiting and in service at department m, and defining the state of the system as the vector $(n_1, n_2, \ldots, n_M)$, he proves the following

THEOREM: Define $P_n^{(m)}$ ($m = 1, 2, \ldots, M$, $n = 0, 1, 2, \ldots$), the Pr [finding n customers in department m in the steady state], by the following equations (where the $P_0^{(m)}$ are determined by the conditions $\sum_n P_n^{(m)} = 1$):

$$P_n^{(m)} = \begin{cases} P_0^{(m)} \left(\Gamma_m / N_m \mu_m \right)^n N_m^{\,n}/n! & (n = 0, 1, \ldots, N_m) \\ \\ P_0^{(m)} \left(\Gamma_m / N_m \mu_m \right)^n (N_m)^{N_m}/N_m! & n \geq N_m \end{cases} \tag{2}$$

A steady state distribution of the state of the above described system is given by the products

$$P\,(n_1, n_2, \ldots, n_M) = P_{n_1}^{(1)} P_{n_2}^{(2)} \ldots P_{n_M}^{(M)} \tag{3}$$

provided $\Gamma_m < \mu_m N_m$ for $m = 1, 2, \ldots, M$

This theorem says, in essence, that at least so far as steady states are concerned, the system with which we are concerned behaves as if its departments were independent elementary systems of the following type (which is the type considered by Erlang): Customers arrive in a Poisson type time series at mean rate λ. They are handled on a first come, first serve basis by a system of N identical servers, the servicing times being exponentially distributed with mean $1/\mu$. The steady state distribution of the number of people, n, waiting and in service has been obtained by Erlang, and is the identical form as in Jackson's theorem above, with $N_m = N$, $\Gamma_m = \lambda$, $\mu_m = \mu$, $P_n^{(m)} = P_n$, and with the condition $\lambda < \mu N$. That is, Jackson's problem reduces to that of Erlang's when $M = 1$. However, for $M = 1$, the network property of the system is destroyed. Jackson's result is very neat, and suggests the possibility of being able to handle large nets of the type of interest to this thesis.

Following, is a statement and discussion of some results obtained for systems similar to those considered by Erlang and Jackson; proofs for the theorems are given in the Appendix.

Consider a pair of nodes in a large communication net. When the first of these nodes transmits a message destined for the other, one can inquire as to what the rest of the net appears like, from the point of view of the transmitting node. In answer to this inquiry, it does not seem unreasonable to consider that the rest of the net offers, to the message, a number N, of "equivalent" alternate paths from the first node to the second; the equivalence being a very gross simplification of the actual situation, which, nevertheless, serves a useful purpose. Thus, the system under consideration reduces itself to that considered by Erlang. Now, for given conditions of average traffic flow and total transmitting capacity between the two nodes, the problem as to the optimum value of N presents itself (optimum here referring to that value of N which minimizes the total time spent in the transmitting node, i.e., time spent waiting for a free transmission channel plus time spent in transmitting the message). Thus, as shown in Figure 1, the system consists of N channels, each of capacity C/N bits per second, with Poisson arrivals of mean rate λ, and with the message lengths distributed exponentially with mean length $1/\mu$ bits.

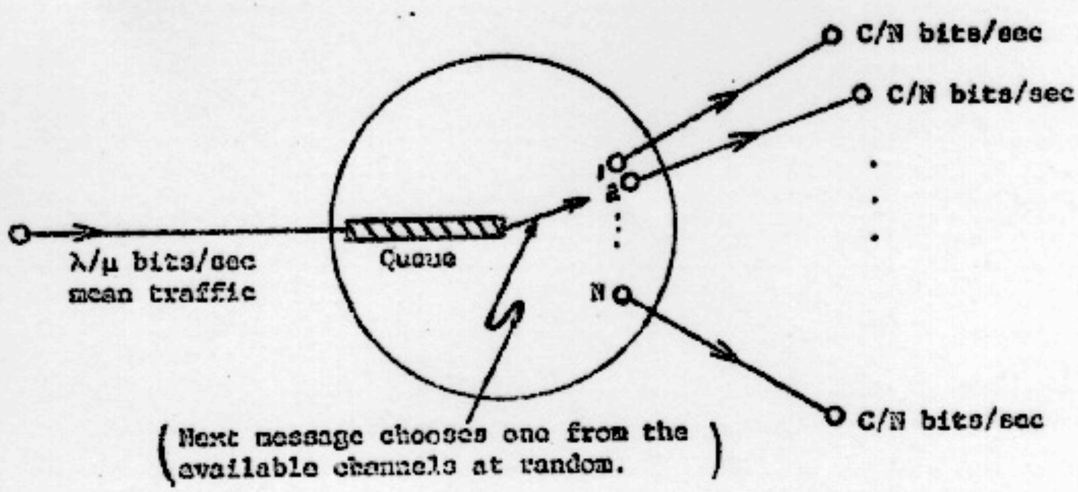

Figure 1: N-channel node considered in Theorem 1.

As is well known, the solution for P_n (defined as the probability of finding n messages in the system in the steady state) is, for $\lambda/\mu C < 1$,

$$P_n = \begin{cases} P_0 p^n N^n / n! & n \leq N \\ P_0 p^n N^N / N! & n \geq N \end{cases} \tag{4}$$

where $p = \lambda/\mu C$ is defined as the utilization factor. Note that this is the same solution found by Erlang. From these steady state probabilities, we can easily find $E(t)$, which is the expected value of the time spent in the system, as

$$E(t) = N/\mu C + P\,(\geq N)\ /\ \mu C(1-p) \tag{5}$$

where

$$P(\geq N) = P_0 (Np)^N\ /\ (1-p)N!$$

and

$$P_0 = [\sum_{n=0}^{N-1} (Np)^n/n! + (Np)^N/(1-p)N!]^{-1}$$

We are now ready to state

THEOREM 1:

The value of N which minimizes $E(t)$, for all $0 \leq p < 1$ is $N = 1$.

Let us look at the expression for $E(t)$ a little closer. Note that the quantity $N/\mu C$ is merely the average time spent in transmitting the message over the channel, once a channel is available. Also, $P(\geq N)$ is the probability that a message is forced to enter the queue. Now, from the independence of the messages, one would expect $E(t)$ to be

E(t) = average time spent in channel + average time spent in queue.

Equation (5) is of the form

E(t) = average time spent in channel + (probability of entering the queue)T

where $T = 1/(1-p)\mu C$.

The physical interpretation of the quantity T is that it is the average time spent in the queue, given that a message will join the queue. The interesting thing here is that the quantity T is independent of N.

Let us now recall one of the basic assumptions of Jackson's theorem, namely, that upon completing service in department m, a customer goes immediately to department k with probability π_{km}. If, now, we consider a communication network of nodes and links (channels), it is not at all obvious how we can route messages in the net so as to satisfy this assumption. That is, how can we design a communication network so that an arbitrary message entering node m will, with probability π_{km} be transmitted over that channel which links node m to node k. Clearly, one way to achieve this is to assign each message, as it enters node m, to the channel linking nodes m and k, with probability π_{km}. However, with such a scheme, there would occur situations in which there were messages in the node waiting on a queue at the same time that some of the channels leading out of the node were idle. It seems reasonable, in some cases at least, to prohibit such a condition. Therefore, restricting the existence of idle channels if there are any waiting messages, we arrive at the following

<u>THEOREM 2</u>:

Given a two channel service facility of total capacity C, Poisson arrivals with mean rate λ, message lengths distributed exponentially with mean length $1/\mu$, and the restriction that no channel be idle if a message is waiting in the queue, then, for an arbitrarily chosen number, $0 \leq \pi_1 \leq 1$ it is <u>not</u> possible to find a queue discipline and an assignment of the two channel capacities (the sum being C) such that

$$\Pr \text{ (entering message is transmitted on the first channel) } = \pi_1$$

for all $0 \leq p < 1$

where $p = \lambda/\mu C$

Thus, this theorem shows that one cannot, in general, make an arbitrary assignment of the probability of being transmitted over a particular channel which remains constant for all p. However, in the proof of this theorem, it is shown that it is possible to find a queue discipline and a channel capacity assignment such that the deviation of this probability π_1 is rather small over the entire range $0 \leq p < 1$.

It is also of interest to note that in the proof of Theorem 2, it is shown that the variation of π_1 is zero over $0 \leq p < 1$ for $\pi_1 = 0, 1/2, 1$. In fact this leads to the following

<u>COROLLARY:</u> For the same conditions as Theorem 2, except allowing N channels, and for $\pi_1 = \pi_2 = \ldots = \pi_N = 1/N$, then it is possible to find a queuing discipline and a channel capacity assignment such that

$$\Pr \text{ (entering message is transmitted over the } i^{th} \text{ channel) } = 1/N \text{ for all } 0 \leq p < 1$$

In proving Theorems 4 and 5, as well as in some other investigations which have been started by the author, the solution to a set of non-linear equations was found to be necessary. As is sometimes possible with such equations, the proper transformation of variables permitted the reduction of these equations to a linear system. This transformation turned out to involve that fundamental quantity p, and thus led to

THEOREM 3:

Consider an N channel service facility of total capacity C, Poisson arrivals with mean rate λ, message lengths distributed exponentially with mean length $1/\mu$, and an arbitrary queue discipline. Define the utilization factor

$$p = \lambda/\mu C$$

Then
$$p = 1 - \sum_{n=0}^{\infty} (\overline{C_n}/C) P_n \tag{6}$$

where $\overline{C_n}$ = Expected value of the unused capacity given n lines in use

and P_n = Pr (finding n messages in the system in the steady state)

provided the system reaches a steady state.

Notice that, in Theorem 3, all information regarding the queue discipline is contained and summarized in the quantity $\overline{C_n}$. This theorem corresponds very nicely with one's intuition, as may be seen by rewriting it as

$$p = 1 - E \text{ (unused normalized capacity)}$$

where the normalization is with respect to the total capacity C. It is clear that this last equation may, in turn be written as

$$p = E \text{ (used normalized capacity)}$$

which says that

$$\lambda/\mu = E \text{ (used capacity)} \tag{7}$$

Now, since the average number of messages entering per second is λ and their average length is $1/\mu$ bits per message, the quantity λ/μ is clearly the average number of bits per second entering the facility. Recall that the condition for the existance of a steady state for this system is

$$\lambda/\mu C < 1$$

Thus, if we have a steady state solution, we are guaranteed that $\lambda/\mu < C$ (which says that the facility can handle the incoming traffic) and so the expected value of the capacity used by this input rate will merely be λ/μ; this is precisely what equation (7) states.

In even the simplest conceivable communications network, it seems reasonable to require that when a message reaches the node to which it is addressed, it should leave the system i.e., it is delivered. However, in the assumptions considered by Jackson, there is no final address associated with each "message" and so, the correspondence between the problem considered by Jackson, and that of interest to this thesis is not as close as one might hope.

Therefore, let us consider a communication network with N + 1 nodes, for which the entering messages have associated with them a final destination (address). Once a message reaches its address, it is dropped from the system immediately. Thus, we are altering the model considered by Jackson only slightly; and in order to keep the rest of the system similar to his, we will consider a completely connected net, with all $\pi_i = 1/N$ (i.e., upon entering a node, a message will be transmitted over a particular channel with probability $1/N$, unless the node which it just entered is its final destination, in which case the message leaves the system with probability one). Note that the corollary to Theorem 2 allows us to define such π_i. For such a system, it turns out that Jackson's results still apply with some slight midifications, as stated in

THEOREM 4:

Consider the completely connected N + 1 node system described above. Let each transmission channel leaving node m have capacity C_m/N. Let the incoming messages entering node m from external sources be Poisson at rate λ_m and let the message lengths be exponentially distributed with mean length $1/\mu$. Further, let t_{mj} be the Pr (message entering node m from its external source has, for a final address, node j). Also define $P_n^{(m)}$ as the probability of finding n messages in node m in the steady state.

Then

$$P_n^{(m)} = \begin{cases} P_0^{(m)} \left(\Gamma_m/\mu C_m \right)^n N^n/n! & (n = 0, 1, \ldots, N) \\ \\ P_0^{(m)} \left(\Gamma_m/\mu C_m \right)^n N^N/N! & (n = N, N+1, \ldots) \end{cases} \tag{8}$$

where

$$\Gamma_m = \lambda_m + \sum_{i \neq m} \Gamma_i \; \alpha_{im}/N \tag{9}$$

and α_{im} = Pr (arbitrary message in node i does not have node m for a final address)

provided $\Gamma_m < \mu C_m$ for all m = 1, 2, ..., M.

This theorem is almost identical to Jackson's theorem, as one might expect. Notice that here, the appropriate definition for the utilization factor for node m is $\rho_m = \Gamma_m/\mu C_m$. The definition of Γ_m as given in Eqn. (9) can be shown to agree with the definition for the average arrival rate of messages at node m (analogous to Jackson's definition in Eqn. (1)). The evaluation of α_{im} involves solving a set of simultaneous equations, as does the evaluation of Γ_m. By way of illustration, the solution for Γ_1 and α_{12} in a three node net follows:

$$\Gamma_1 = \frac{2}{3} \left[2\lambda_1 + \lambda_2 t_{23} + \lambda_3 t_{32} \right]$$

$$\alpha_{12} = \left[2\lambda_1 t_{13} + \lambda_2 t_{23} \right] / (3/2)\, \Gamma_1$$

As already mentioned, P. Burke [17] has shown that in a waiting system with N servers, with Poisson arrivals (mean rate λ) and with exponential holding times (mean holding time for each server = $1/\mu$), the traffic departing from the system is Poisson with mean rate λ, providing the steady state prevails (i.e. provided $p = \lambda/\mu N$ is less than 1). In fact, it is on this basis that Jackson is able to say that his system consists of independent elementary systems; that is, Burke's theorem states that exponential waiting systems (or departments or nodes, as the problem may be defined) always transform Poisson input traffic into Poisson output traffic (with the same mean rates) and thus the departing traffic is not distinguishable from the input traffic. An identical situation exists for the system considered in Theorem 4, and is stated formally in

THEOREM 5:

For the system considered in Theorem 4, all traffic flowing within the network is Poisson in nature, and, in particular, the traffic transmitted from node m to any other node in the system in Poisson with mean rate Γ_m/N.

Many of the theorems presented here are fairly specialized to particular conditions on the network topology and on the routing discipline. It is anticipated that a number of them can be extended to less restrictive networks, and such an effort is now being undertaken by the author, since this investigation fits very well with the general aims of the thesis research.

APPENDIX

Proof of Theorems

Before we proceed with the proofs, let us derive a general result for a class of Birth-Death Processes [12].

Let[4]

$P_n(t) = \Pr$ [finding n members in system at time t]

$b_n dt = \Pr$ [birth of a new member during any interval of length dt | n members already in system]

$d_n dt = \Pr$ [death of a member during any interval of length dt | n members in system]

then, clearly

$$P_0(t + dt) = P_1(t)\ (d_1 dt) + P_0(t)\ (1-b_0 dt)$$

$$P_n(t + dt) = P_{n+1}(t)\ (d_{n+1} dt) + P_{n-1}(t)\ (b_{n-1} dt) + P_n(t)\ (1-d_n dt - b_n dt) \qquad n \geq 1$$

From these eqns., we get

$$dP_0(t)/dt = d_1 P_1(t) - b_0 P_0(t) \qquad \text{(A1)}$$

$$dP_n(t)/dt = d_{n+1} P_{n+1}(t) + b_{n-1} P_{n-1}(t) - (d_n + b_n) P_n(t) \qquad n \geq 1 \qquad \text{(A2)}$$

Let us now assume the existence of a steady state distribution for $P_n(t)$, that is,

$$\lim_{t \to \infty} P_n(t) = P_n$$

Therefore $\displaystyle \lim_{t \to \infty} dP_n(t)/dt = 0$

and so, we get, for eqns. (A1) and (A2),

$$0 = d_1 P_1 - b_0 P_0$$

$$0 = d_{n+1} P_{n+1} + b_{n-1} P_{n-1} - (d_n + b_n) P_n \qquad n \geq 1$$

---

[4] Note that b_n and d_n are assumed to be independent of time.

The solution to this set of difference equations is

$$P_n = \prod_{i=0}^{n-1} P_0(b_i/d_{i+1}) \qquad n \geqslant 1 \qquad \text{(A3)}$$

which may easily be checked.

<u>Theorem I - proof:</u>

Given $$E(t) = N/\mu C + P(\geqslant N)/\mu C(1-p)$$

substituting for $P(\geqslant N)$ and rearranging terms gives us

$$E(t) = (N/\mu C)\ [1 + \frac{1/N(1-p)}{S_N(1-p) + 1}\ ] \qquad \text{(A4)}$$

where $S_N = \sum_{n=0}^{N-1} (Np)^{n-N} N!/n! \ > \ 0$

now $S_N = \sum_{n=0}^{N-1} p^{n-N}\ [N/N]\ [(N-1)/N]\ \ldots\ [(n+1)/N]$

therefore $S_N \leqslant \sum_{n=0}^{N-1} p^{n-N} = (p^{-N}-1)/(1-p)$

giving $$0 < S_N \leqslant \frac{p^{-N}-1}{1-p} \qquad \text{(A5)}$$

Now, for $N = 1$, eqn. (A4) yields

$$E(t) = 1/\mu C(1-p) \qquad \text{for } N = 1$$

therefore, it is sufficient to show that

$$E(t) > 1/\mu C(1-p) \text{ for all } N > 1,\ \ 0 \leqslant p < 1$$

using (A5) we get, for (A4)

$$E(t) \geqslant (N/\mu C)\ [1 + p^N/N(1-p)]$$

$$E(t) \geqslant [N(1-p) + p^N]/\mu C(1-p)$$

Letting $1-p = \alpha$ or $1-\alpha = p$, we see that

$$N(1-p) + p^N = N\alpha + (1-\alpha)^N \geq N\alpha + 1 - N\alpha = 1$$

thus $\quad E(t) \geq 1/\mu C(1-p)$

for all N, and in particular, the only case for which the equality holds is $N = 1$. Note that the equality would also hold for $\alpha = 0$ but this implies that $p = 1$, which we do not permit. Thus

$$E(t) > 1/\mu C(1-p) \quad \text{for } N > 1, \; 0 \leq p < 1$$

which completes the proof. ∎

Theorem 2 - proof:

The method of proof will be to show the impossibility of contradicting the theorem.

Suppose $p \to 0$. Then P_0 (the probability that in the steady state the system is empty) approaches 1. In such a case, an entering message (which will, with probability arbitrarily close to 1, find an empty system) must be assigned to channel 1 with probability π_1 (and to channel 2 with probability $\pi_2 = 1-\pi_1$) if one is to have any hope of contradicting the theorem.

Now suppose $p \to 1$; then P_0 and P_1 (the probability of one message in the system) both approach 0. Therefore, the channel capacity C_1 assigned to channel 1 (which implies $C-C_1 = C_2$ for channel 2) must be chosen so that

$$\alpha = \Pr\,[\text{channel 1 empties before channel 2} \mid \text{both channels busy}] = \pi_1$$

That is, with probability arbitrarily close to 1, a message entering the node will be forced to join a queue, and so, when it reaches the head of the queue, it will find both channels busy. If this message is to be transmitted over channel 1 with probability π_1, it must be that the channel capacity assignments result in $\alpha = \pi_1$. Note that we have taken advantage of the fact that messages with exponentially distributed lengths exhibit no memory as regards their transmission time.

Now, $\alpha = \int_{t=0}^{t=\infty}$ Pr [channel 1 empties in $(t, t+dt)$ | both busy at time 0]
$\cdot$Pr [channel 2 is not yet empty by t | both busy at time 0]

$$\alpha = \int_0^\infty \mu C_1 e^{-\mu C_1 t} e^{-\mu C_2 t} dt$$

$$\alpha = \mu C_1/(\mu C_1 + \mu C_2) = C_1/C$$

but

$$\alpha = \varkappa_1$$

therefore

$$C_1 = \varkappa_1 C$$

and also

$$C_2 = \varkappa_2 C = (1 - \varkappa_1)C$$

These two limiting cases for $p \to 0$ and $p \to 1$ have constrained the construction of our system completely.

Now, let

r_1 = Pr [incoming message is transmitted on channel 1]

P_n = Pr [finding n messages in the system in the steady state]

Then clearly,

$$r_1 = \varkappa_1 P_0 + q_{21} P_1 + \sum_{n=2}^{\infty} \varkappa_1 P_n \tag{A6}$$

where

q_{i1} = Pr [channel i is busy | only one channel is busy]

that is

r_1 = E [probability of an arbitrary message being transmitted over channel 1]

For q_{21}, we write:

$$q_{21}(t+dt) = [P_0(t)/P_1(t)](\lambda\varkappa_2 dt) + [P_2(t)/P_1(t)](\mu C_1 dt) + q_{21}(t)[1-\lambda dt-\mu C_2 dt]$$

Assuming a steady state distribution, we get,

$$0 = (P_0/P_1)\lambda\varkappa_2 + (P_2/P_1)\mu C_1 - (\lambda+\mu C_2)q_{21} \tag{A7}$$

Now, since this system satisfies the hypothesis of the Birth-Death Process considered earlier, we apply Eqn.(A3), with $d_1 = \mu\overline{C}$, $d_n = \mu C$ $(n \geq 2)$, $b_n = \lambda$, and obtain

$$P_n = \begin{cases} (C/\bar{C})p^n P_0 & n \geq 1 \\ P_0 & n = 0 \end{cases} \qquad \text{(A8)}$$

where $p = \lambda/\mu C$,

and $\bar{C} = E$ [capacity in use | one channel is busy]

$$= \mu C_1 q_{11} + \mu C_2 q_{21}$$

Also, recall that $C_1 = \alpha_1 C$ and $C_2 = \alpha_2 C = (1-\alpha_1)C$.
Thus, Eqn.(A7) becomes

$$q_{21} = (\mu\bar{C}\alpha_2 + \lambda\alpha_1) \;/\; (\lambda + \mu C\alpha_2) \qquad \text{(A9)}$$

similarly

$$q_{11} = (\mu\bar{C}\alpha_1 + \lambda\alpha_2) \;/\; (\lambda + \mu C\alpha_1)$$

Now, forming the equation,

$$q_{11} + q_{21} = 1$$

we obtain, after some algebra,

$$\mu\bar{C} = \mu C(\mu C + 2\lambda) \;/\; (2\mu C + \lambda/\alpha_1\alpha_2)$$

$$= \mu C(1 + 2p) \;/\; (2 + p/\alpha_1\alpha_2)$$

We may now write Eqn.(A9) as

$$q_{21} = \frac{[\mu C(1+2p)/(2+p/\alpha_1\alpha_2)]\alpha_2 + \lambda\alpha_1}{\lambda + \mu C\alpha_2}$$

Simplifying, we get

$$q_{21} = \alpha_1(\alpha_2+p) \;/\; (2\alpha_1\alpha_2+p) \qquad \text{(A10)}$$

Returning to Eqn.(A6), we see that the only way in which r_1 can equal α_1 is for $q_{21} = \alpha_1$. Eqn. (A10) shows that this is not the case, which demonstrates that the theorem cannot be contradicted for an arbitrary α_1 , proving the theorem. ∎

However, it can be seen from Eqn. (A10), that $q_{21} = x_1$ for $x_1 = 0,\ 1/2,\ 1$ only. Let us now form r_1 from Eqns. (A6) and (A8):

$$r_1 = P_0 [\, x_1 + (\lambda q_{21}/\mu \bar{C}) + (x_1 C/\bar{C}) \sum_{n=2}^{\infty} p^n]$$

where P_0 is found from Eqn. (A8) by requiring

$$\sum_{n=0}^{\infty} P_n = 1$$

After substituting and simplifying, we get

$$r_1 = x_1 \left[\frac{x_1 p^2 + (1-x_1{}^2)p + x_1 x_2}{(1-2x_1 x_2)p^2 + 3x_1 x_2 p + x_1 x_2} \right]$$

Figure (A-1) shows a plot of r_1 as a function of p, with x_1 as a parameter.

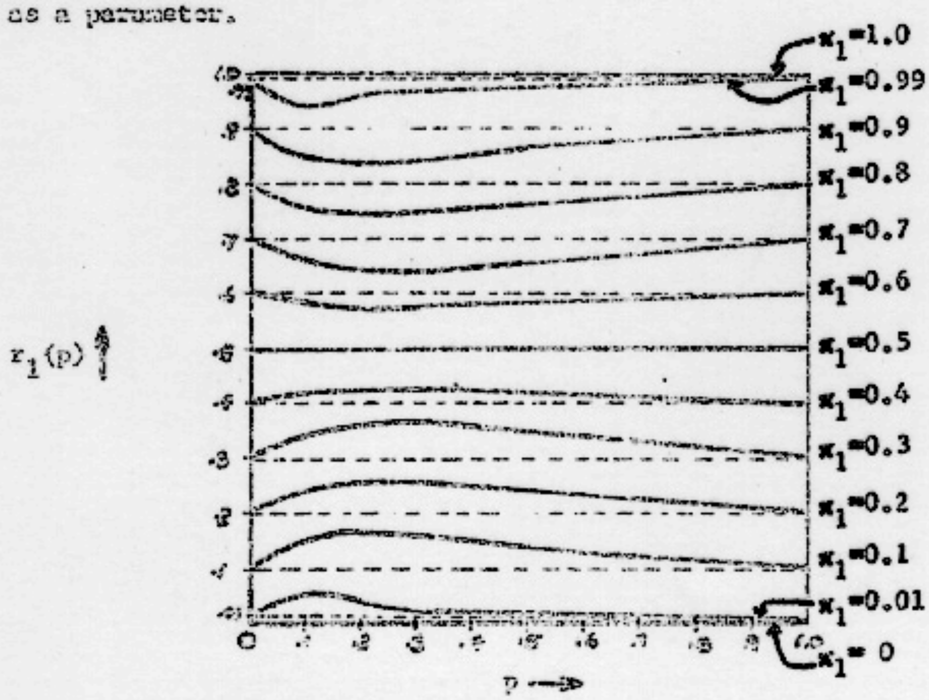

Figure A-1: Variation of r_1 with p.

Note that the variation of r_1 is not too great. This illustrates that although Theorem 2 is written as a negative result, its proof demonstrates a positive result, namely, that the variation of r_1 is not excessive. The arrangement which gives this behavior is one in which the channel capacity is divided between the two channels in proportion to the desired probability of using each channel, and for the discipline followed when a message finds both channels empty, one merely chooses channel i with probability κ_i.

Corollary to Theorem 2 - proof:

In proving Theorem 2, it was shown that for $\kappa_1 = 1/2$, a suitable system could be found to realize this κ_1. This result also follows directly from the complete symmetry of the two channels. Similarly, the proof of this corollary follows trivially from recognizing, once again, the complete symmetry of each of the N channels. ∎

Theorem 3 - proof:

The system considered in this theorem satisfies the conditions of the Birth-Death Process examined earlier, with

$$b_n = \lambda$$
$$d_n = \mu(C-\overline{C_n})$$

Thus, by Eqn. (A3), we find

$$P_n = P_0(\lambda/\mu)^n \;/\; [\prod_{i=1}^{n} (C-\overline{C_i})] \qquad n \geq 1$$

or

$$P_n = P_0 p^n \;/\; [\prod_{i=1}^{n} (1-r_i)] \qquad n \geq 1 \qquad \text{(A11)}$$

where $p = \lambda/\mu C$

$r_i = \overline{C_i}/C$ (A12)

and $P_n = P_0$ for n=0, by definition.

Let us now solve for P_0:

$$\sum_{n=0}^{\infty} P_n = 1 = P_0[\, 1 + \sum_{n=1}^{\infty} R_n p^n \,]$$

where

$$R_n = 1 \;/\; \prod_{i=1}^{n} (1-r_i)$$

Thus $$P_0 = 1 / [1 + \sum_{n=1}^{\infty} R_n p^n] \tag{A13}$$

Now, according to the statement of the theorem, let us form and solve for

$$x = 1 - \sum_{n=0}^{\infty} (\overline{C_n}/C)P_n$$

Noting that $\overline{C_0} = C$ by construction, and using Eqns. (A11) - (A12),

$$x = 1 - P_0 - P_0 \sum_{n=1}^{\infty} r_n R_n p^n$$

$$x = 1 - \frac{1 + \sum_{n=1}^{\infty} r_n R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

$$x = \frac{1 + \sum_{n=1}^{\infty} R_n p^n - 1 - \sum_{n=1}^{\infty} r_n R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

$$x = \frac{\sum_{n=1}^{\infty} R_{n-1} p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

$$x = \frac{p \sum_{n=0}^{\infty} R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

It is important to recognize here that R_0 must be defined as

$$R_0 = (1-r_1)R_1$$

Thus

$$R_0 = 1 \qquad \text{(taken now as a definition as well)}$$

and so

$$x = p \frac{1 + \sum_{n=1}^{\infty} R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

or $\quad x = p$

which proves the theorem.

<u>Theorem 4 - proof:</u>

The system considered in this theorem satisfies the conditions of the Birth-Death Process examined earlier. However, we have N+1 nodes, and so we must investigate N+1 probability distributions, $P_n^{(m)}$, where $m = 1,2,\ldots,N+1$ and $n = 0,1,2,\ldots$ Let the birth and death rates for node m be $b_n^{(m)}$ and $d_n^{(m)}$ respectively. With this notation, we see that

$$b_n^{(m)} = \lambda_m + \sum_{\substack{j=1 \\ j \neq m}}^{N+1} (\mu C_j/N)\alpha_{jm}\left[1 - \sum_{i=0}^{N-1} P_i^{(j)} (N-i)/N\right] \qquad n \geq 0$$

$$d_n^{(m)} = \begin{cases} n\mu C_m/N & n = 0,1,\ldots,N \\ \mu C_m & n \geq N \end{cases}$$

An explanation of the $b_n^{(m)}$ is required at this point. λ_m is the input (birth) rate of messages to node m from its external source (by definition). In addition, each of the other N nodes sends messages to node m. Let us consider the j^{th} node's contribution (x_j say) to the input rate of node m ($j \neq m$):
Clearly,

$$x_j dt = Pr[Q_1, Q_2, Q_3] = Pr[Q_1 \mid Q_2, Q_3]\ Pr[Q_2, Q_3]$$

where Q_1 = event that a message on the channel connecting node j to node m completes its transmission to node m in an arbitrary time interval (t, t+dt)

Q_2 = event that an arbitrary message in node j does not have node m for its final address

Q_3 = event that the channel connecting node j to node m is being used

Since Q_2 and Q_3 are independent events, we get

$$x_j dt = \Pr[Q_1 \mid Q_2, Q_3] \Pr[Q_2] \Pr[Q_3]$$

and for node j,

$$\Pr[Q_1 \mid Q_2, Q_3] = \Pr[Q_1 \mid Q_3] = (\mu C_j / N) dt$$

$$\Pr[Q_2] = \alpha_{jm}$$

$$\Pr[Q_3] = 1 - \sum_{i=0}^{N-1} p_i^{(j)} (N-i)/N$$

The summation on j appearing in the expression for $b_n^{(m)}$ merely adds up the contributions to the input (birth) rate of internally routed messages.

Now, according to the definition of $\overline{C_n}$ in Theorem 3, we can apply the same definition to each of the N+1 nodes in the present theorem. Thus, in this case, we see that

$$\overline{C}_n^{(m)} = \begin{cases} [(N-n)/N]C & n \leqslant N \\ 0 & n \geqslant N \end{cases}$$

and so we recognize, by application of Theorem 3, that

$$1 - \sum_{i=0}^{N-1} p_i^{(j)} (N-i)/N = p_j$$

where, by definition,

$$\rho_j = \frac{\Gamma_j/\mu}{C_j} = \frac{\text{average input rate to node } j}{\text{maximum output rate from node } j}$$

$$\text{thus } b_n^{(m)} = \lambda_m + \sum_{\substack{j=1 \\ j \neq m}}^{N+1} \Gamma_j \alpha_{jm}/N \qquad \text{all } n \geq 0$$

But, since $b_n^{(m)}$ is independent of n, it obviously satisfies the definition of Γ_m (= average number of messages per second entering node m). Thus

$$b_n^{(m)} = \Gamma_m = \lambda_m + \sum_{\substack{j=1 \\ j \neq m}}^{N+1} \Gamma_j \alpha_{jm}/N$$

Now, using these birth and death coefficients, we apply eqn. (A3) to get

$$p_n^{(m)} = \begin{cases} p_0^{(m)} (\Gamma_m/\mu C_m)^n N^n/n! & (n = 0, 1, \ldots, N) \\ \\ p_0^{(m)} (\Gamma_m/\mu C_m)^n N^N/N! & n \geq N \end{cases}$$

In this case, it is clear that the steady state is defined only when

$$\rho_m = \Gamma_m/\mu C_m < 1 \qquad \text{for all } m = 1, 2, \ldots, M$$

This completes the proof of the theorem. ∎

Theorem 5 - proof:

In order to show that all traffic flowing within the network considered in Theorem 4, is Poisson in nature, it is sufficient to show that

$$q\ (C,t) = \Pr\ \{\text{a message transmission, in any channel } C \text{ of the network, is completed in a time interval } (t,\ t + dt), \text{ where } t \text{ is arbitrary}\}$$
$$= k dt$$

where k is a constant.

Let us show this for an arbitrary channel connecting node j (say) to any other node:

$$q\ (C_j/N,t) = \Pr\ [Q_1 \mid Q_3]\ P\ [Q_3]$$

where Q_1 and Q_3 are as defined in the proof of Theorem 4. As shown in the proof of Theorem 4,

$$\Pr [Q_1 \mid Q_3]\ \Pr\ [Q_3] = (\mu C_j/N)\ [1 - \sum_{i=0}^{N-1} P_i^{(j)} (N-i)/N]\ dt$$

$$= (\mu C_j/N) p_j dt$$

and so $q(C_j/N,t) = (\Gamma_j/N)\ dt$

which proves the theorem, and also shows the value of the mean rate for the Poisson traffic to be Γ_j/N. $\blacksquare$

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