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# Maximum Probability of Successful Transmission in a Random Planar Packet Radio Network*

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*Abstract* — **Suppose a packet radio network has nodes which are randomly distributed over the infinite plane according to a Poisson point process such that nodes have an average of $N$ nodes within its transmission range. In this paper we show that over all protocols, the maximum probability of a successful transmission over any period of time is upper bounded by $.9278/N$ for suitably large $N$. We compare this performance to that obtained using slotted-ALOHA and CSMA, and show, for realistic networks, that these protocols at best achieve respectively about 36% and 49% of our bound.**

## 1 Introduction

In this paper we assume we are given a connected, random, network in which the average number of neighboring nodes is given by $N$ (assumed to be fixed), and seek to determine the maximum number of simultaneous transmissions in the network that can be successful. This number, divided by the number of nodes in the network, will give the maximum probability of successful transmission and will be an upper bound for any random access protocol that does not use information about the directionality or locations of neighboring nodes. This result can be used then as a standard to evaluate the performance of other protocols in this environment.

## 2 Model and Analysis

We will assume that nodes of a packet radio network are distributed on the plane according to a Poisson point process with a mean density of $\lambda$ radio units per unit area. With no loss of generality we will assume that radios transmit with a range of one unit ($R=1$). The average number of neighbors they have, $N$, therefore, is given by $N=\lambda\pi$. From any given node, $a$, another node, $b$, is said to be $i$-hops away if there exists a path from $a$ to $b$ that contains $i-1$ other nodes and no other path exists between $a$ and $b$ that contains fewer nodes. In our derivation we will focus on a section of the network containing $n$ nodes, and thus will be concerned with a portion of the network of radius $R_0$, where $n=\lambda\pi R_0^2$. Our final results are independent of $n$. We will assume the network formed by these $n$ nodes is connected. We should mention here that our network model is that of a snapshot of a mobile packet radio network (thus the random distribution of nodes on the plane). In such networks it is extremely difficult for nodes to ascertain the locations of other nodes in the network, and, in particular, nodes do not know the direction of the recipients of their transmitted packets. Our results here are applicable only to protocols that do not utilize information concerning the location or direction of their neighboring nodes.

To motivate how we propose to calculate our upper bound suppose a node, $a$, is transmitting a packet to one of its neighbors. Let $S_i$ be the set of nodes that are $i$-hops away from $a$, and define a *k-order independent set* to be a set of nodes that are all mutually $k$ or more hops away from each other. A *maximal k-order independent set* is a $k$-order independent set to which no other node of the network can be added. It is clear that if all nodes of a maximal 3-order independent set transmit, and these are the only transmitting nodes in the network, all of their transmissions will be *successfully received*, where by this we mean that nodes in the network that are possible recipients of any of these messages, hear exactly one transmitter. An example of such a set is shown in figure 1. In this figure, nodes of the 3-order independent set are indicated by the larger circles. We can easily show the following property of maximal 3-order independent sets:

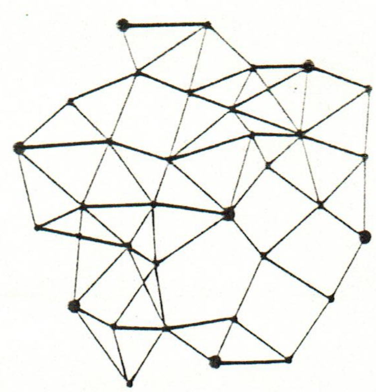

FIGURE 1
A maximal 3-order independent set.

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*Lemma:* For any maximal 3-order independent set $M$, no node not in the set can transmit without causing interference with a transmission of at least one node in $M$.

*Proof:* Suppose $a$ was such a node. Using the previous definitions, it is clear that $S_1 \cap M = \varnothing$ (where $\varnothing$ is the null set) otherwise $a$ would be interfering with its own reception from nodes in $M$. If $S_2 \cap M \neq \varnothing$ then this implies there is a node that simultaneously receives signals from $a$ and also from a node in $M$. Thus $S_2 \cap M = \varnothing$, but this implies that $M$ is not maximal since $a$ is at least 3 or more hops away from every node in $M$.

We should observe that this lemma is also true for maximal $k$-order independent sets where $k = 4, 5$, and thus we cannot immediately conclude that a maximal 3-order independent set corresponds to the greatest number of transmissions in the network that are guaranteed to cause no collisions. Intuitively, however, to achieve maximal throughput we would want transmitters to be as close to each other as possible, without having collisions detract from the throughput of the channel. To make this more precise, let $S_k$ be the set of all maximal $k$-order independent sets, and let $L_k$ be the cardinality of the largest set in $S_k$ (we are assuming here a finite but arbitrarily large graph). Then, since a $k+1$-order independent set is also a $k$-order independent set, we have that $L_k \geqslant L_{k+1}$. Thus, since collisions occur for $k = 1, 2$, we can conclude that the largest number of successful transmissions, without allowing collisions, is given by $L_3$.

Since nodes in this scenario that are allowed to transmit are mutually at least three hops away from each other and at best exactly three hops, in the ideal case we can imagine the plane being tessellated with equilateral triangles having sides equal to the average distance between nodes three hops away. This tessellation is motivated in figure 2 where we have connected the nodes of the maximal 3-order set of figure 1 (not all such configurations will result in a hexagonal shaped figure).

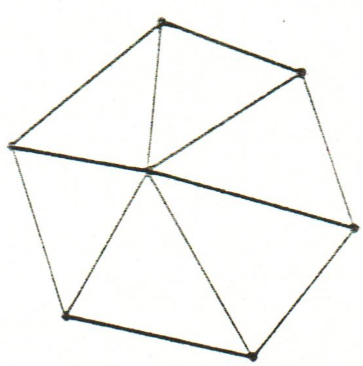

FIGURE 2
Tessellation of Figure 1.

For such a tessellation each vertex corresponds to a transmitting node of the network. The number of such triangles, for a given section of the network, will correspond to twice the number of vertices. This can be seen in figure 3 where we have mapped each vertex to the triangle lying directly above it (shown by the arrows in the figure).

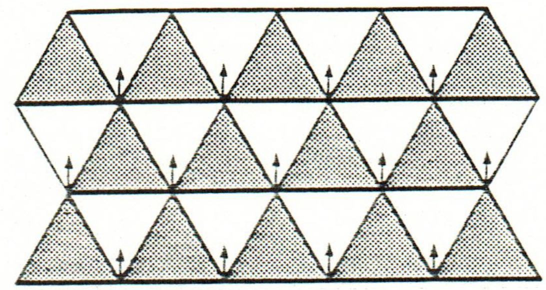

FIGURE 3
Mapping of vertices to triangles.

The shaded triangles have no corresponding vertices and can then be seen to be equal in number to those that are mapped to vertices. If we let $\bar{X}$ be the average distance between nodes three hops away, then the area of each triangle is given by $\sqrt{3}\,\bar{X}^2/4$. Letting $T$ be the number of triangles in network under consideration whose total area equals $\pi R_0^2$, we can write:

$$T = \pi R_0^2 / (\sqrt{3}\,\bar{X}^2/4)$$

The fraction $f$ of successful transmissions then can be written as:

$$f = T/2n = \frac{2}{\lambda\sqrt{3}\,\bar{X}^2}$$

In [1] it is shown that $\bar{X} = 2$ and thus we have (using the fact that $N = \lambda\pi$):

$$f(N) = .9068/N \qquad (1)$$

This then represents the fraction of successful transmissions that occur in a network using a protocol that schedules transmissions in a manner that at all times a maximum number of nodes in the network transmit and there is no possibility for collisions. To check the intuition that lead to this equation we generated random planar connected graphs with different mean densities. These graphs had from 50 to 90 nodes. Using these graphs we found the size of the maximal 3-order independent set. The fraction of these nodes was then calculated. In Figure 4 we have plotted $f(N)$ as well as these generated values and we see a close match. The data that lies above the $f(N)$ curve is a result of the edge-effects obtained from generating finite graphs. In such graphs, nodes at the edge of the graph are more likely to belong to maximal independent sets since they have neighbors only on one side. This tends to increase the size of the maximal 3-order independent sets.

It is interesting to graphically see what $\bar{X} = 2$ means in terms of transmission areas. We see in figure 5 that this average third hop distance implies that the plane is covered with unit circles from transmitters located at the vertices of the equilateral triangles. Observe that there is no overlap of the circles and that very little of the plane is not covered. In fact it is well known that this arrangement of unit circles maximizes the density of the area of the plane that is covered by only one circle (called singly covered) provided that no area is covered twice [2]. Viewing the problem in this manner leads us to inquire about the arrangement,

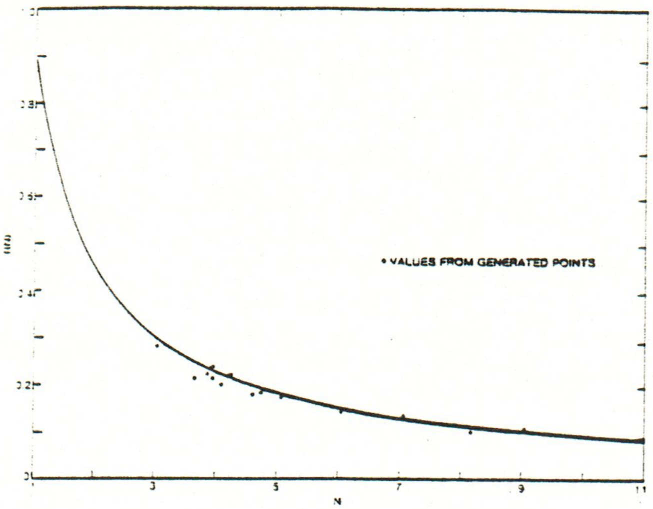


FIGURE 4
Comparing $f(N)$ with Generated Data.

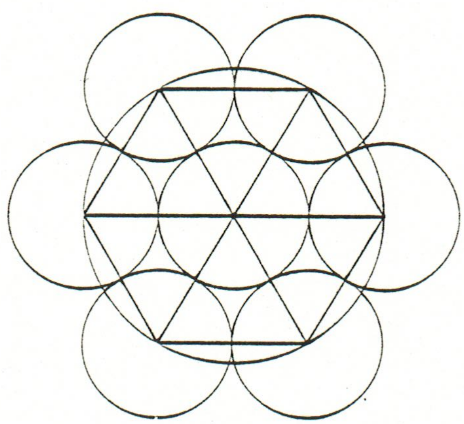

FIGURE 5
$\bar{X}$ in terms of transmission areas.

allowing possible overlap of circles, that yields the highest density of singly covered area. This density will yield an upper bound to the fraction of the expected number of successful transmissions for a random packet radio network, since the probability that a transmission is successfully received is equal to the fraction of area that is covered by only its transmission. We can use figure 6 to this end. This figure is a reproduction of one of the triangles of figure 5, where we have drawn the triangles with a length less than 2 units.

In this figure let $I(x)$ be the area of overlap within the triangle, and let $E(x)$ be the area which is not covered in the circle tessellation. First let us derive equations for these two functions. We can use the equation derived in the appendix for $A(\cdot)$ to write:

$$I(x) = (\pi - A(x,1))/2$$

which after some manipulation becomes:

$$I(x) = \cos^{-1}(x/2) - \frac{x}{2}\sqrt{1-\frac{x^2}{4}}$$

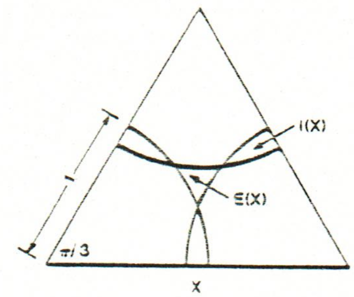


FIGURE 6
Section of Plane with Overlapped Tessellation.

The amount of area within the triangle that is covered is $C(x) = \pi/2 - 3I(x)$ and subtracting this from the area of the triangle gives:

$$E(x) = \frac{\sqrt{3}}{4}x^2 - \frac{\pi}{2} + 3\cos^{-1}(x/2) - \frac{3x}{2}\sqrt{1-\frac{x^2}{4}}$$

Now suppose we wanted to "partially" tessellate the plane with unit circles in such a way that the sum of the overlapped and the uncovered areas was minimized (thus maximizing the amount of singly covered area). It is clear from figure 6, that $I(x)$ is increasing in $x$ and $E(x)$ is decreasing, and thus seeking a minimum is a well formed problem. Letting the objective function be $F(x)$ we then have:

$$F(x) = \frac{\sqrt{3}}{4}x^2 - \frac{\pi}{2} + 6\cos^{-1}(x/2) - 3x\sqrt{1-\frac{x^2}{4}}$$

This function is minimized for $x^* = 1.9215$ which gives the tessellation shown in figure 7 (the points of intersection form a 12-gon). We can also derive the fraction of the plane that is covered by this type of tessellation by forming the ratio of the covered area to the area of the triangle. When this is done, the fraction is determined to be .9278, (as compared to .9068 of equation (1)) and thus about 92% of the plane is singly covered. We can thus write the upper bound as:

$$f^*(N) = .9278/N \qquad (2)$$

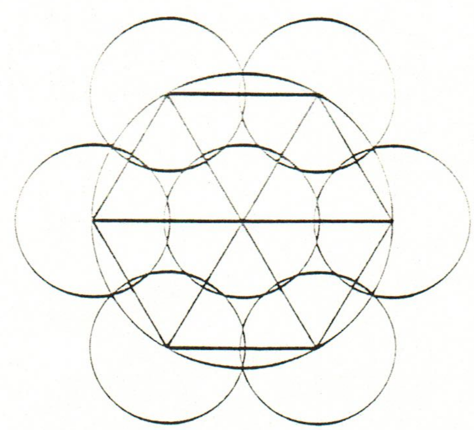

FIGURE 7
Allowing Collisions to Increase the Bound.

This arrangement of circles on the plane was conjectured to have the highest density of singly covered area in 1964 [3] and has recently been proved by Ernst Strass at UCLA (unpublished manuscript). We should observe that for reasonably large $N$, equations (1) and (2) agree for practical purposes and thus all the approximations and conclusions made using equation (1) are also applicable to our new bound.

If we let $G$ be the offered load (the average number of packets presented to the channel per unit time) of traffic coming from the area of one circle, we clearly see that $G \cong 1$. This corroborates similar results in other models [4, 5] where it was found that $G \approx 1$ for slotted-ALOHA networks. This result then is a generalization of that for a single-hop network where it can be shown that an offered load of one packet per unit time optimizes performance [6]. We can derive a rule of thumb for multi-hop packet radio networks by approximating equation (2) as:

$$f^*(N) \cong 1/N$$

It can best be interpreted by looking at figure 7. Here we see that the plane can also be viewed as being almost tessellated by circles of unit radius, the center of which contains a transmitter. Since each transmitter sends to an average of $N$ other nodes, it is clear that only one of $N$ nodes will be successful, thus giving the approximate $1/N$ ratio.

## 3 Comparison to slotted-ALOHA

We can compare this optimal fraction of transmissions to that of the slotted-ALOHA radio network studied in [5]. In that work, the equation for the probability of successful transmission in the non-capture case, (which corresponds to our $f^*(N)$) was given by:

$$P(N,p) = (1-p)p(1-e^{-N/2})e^{-Np} \tag{3}$$

where $p$ is the probability of transmitting in any randomly selected slot. The $(1-e^{-N/2})$ term in this equation is a factor that represents the probability that the network is connected. Since our derivation of $f^*(N)$ assumes a connected graph, we should eliminate the connectivity term from equation (3) before comparing the performance of this system to that of the optimal protocol.

For a given $N$, $P(N,p)$ achieves a maximal value for:

$$p^* \triangleq q(N) = \frac{N+2-\sqrt{N^2+4}}{2N}$$

We can thus write the maximum fraction of successful transmissions for the slotted-ALOHA network as:

$$P^*(N) = q(N)(1-q(N))e^{-Nq(N)} \tag{4}$$

The *efficiency* of a protocol is defined to be the ratio of its performance to that of the optimal protocol and thus we define:

$$e(N) \triangleq P^*(N)/f^*(N)$$

In figure 8 we have plotted $f^*(N)$, $P^*(N)$ and $e(N)$ for $1 \leqslant N \leqslant 11$. In this figure we observe that $e(N)$ increases in $N$. To find its limit, we observe:

$$\lim_{N\to\infty} q(N) = 1/N$$

hence we have:

$$\lim_{N\to\infty} e(N) = 1/(.9278\,e) = .396$$

Thus the capacity of slotted-ALOHA at best is about 40% of the channel capacity as given by our upper bound. Observe that the optimal capacity of slotted-ALOHA occurs when the density of terminals approaches infinity. This of course corresponds to an unrealistic network configuration. The increase in efficiency as $N$ grows larger is, however, very slow. For realistic size networks, looking at figure 8, we see that we can approximate the efficiency to be about $1/e$. In a single-hop environment slotted-ALOHA achieves a maximum capacity of $1/e$ and thus we have the intuitively pleasing result that the maximum capacity of the slotted-ALOHA protocol is about $1/e$ in both single and multi-hop environments.

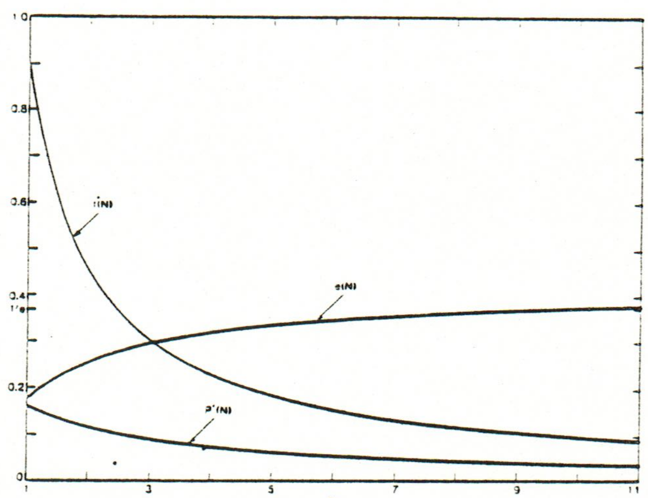


FIGURE 8
The Efficiency of slotted-ALOHA.

## 4 Comparison with CSMA

In this section we will use the method defined in the previous sections to derive an equation for the optimal performance for CSMA in a multi-hop environment and then will calculate the efficiency of this protocol. In a CSMA environment, nodes that hear an idle channel can transmit packets on the channel. In a random connected network the maximal number of nodes that could transmit using the CSMA protocol is equal to the size of the maximal 2-order independent set. We can calculate the fraction of nodes in such a network using methods developed in previous sections. If we imagine, in figure 3, the length of the equilateral triangle to be the average euclidean distance for nodes separated by 2 hops, defined to be $\overline{Y}$, then $g(N)$, the fraction of nodes in a maximal 2-order independent set, can be calculated as:

$$g(N) = 2.214/N$$

where we have used the result derived in [1] that $\overline{Y} = 1.2881$. If we assume the fraction of the nodes in a maximal 2-order independent set that transmit is $p$, then the fraction of transmissions in the network is given by $g(N)p$. Unlike the derivation of the optimal protocol however, in this case not all transmissions are successful. We can calculate the probability that a randomly selected transmitter is successful by using figure 9. In this figure we have shown the triangle tessellation corresponding to a maximal 2-order independent set, and the circular tessellation that results when nodes on the vertices of the triangles transmit.

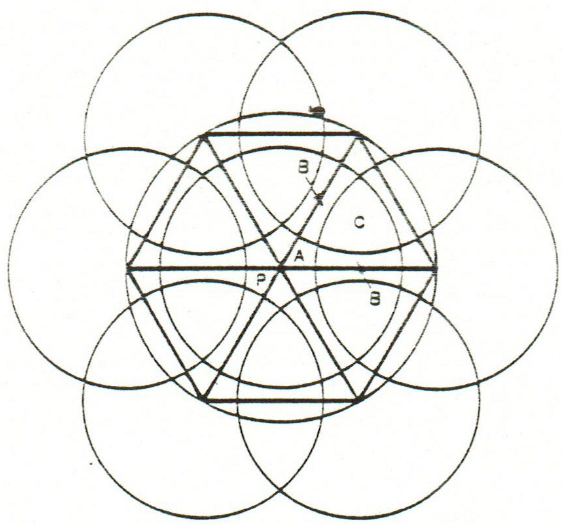


FIGURE 9
Tessellation for CSMA Calculation.

If we suppose in this figure that $P$ is a transmitter, then we can calculate the probability that it is successful by determining the probability that $P$ is sending its message to a neighboring node located in one of the areas labeled $A$, $B$, or $C$ of the figure. We will assume a uniform probability in making this calculation and also assume that each transmission corresponds to a series of messages having infinitesimal length. This assumption allows us to calculate the expected fraction of messages that are successfully received before being collided by an interfering transmission. An alternative way to model this is to assume that each receiver captures the first signal that is sent to it. Although such *perfect time capture* [7] is not attainable in real networks, this assumption will allow us to calculate an upper bound for network performance. Area $C + B$ forms half of the intersection of two unit circles separated by a distance of $\bar{Y}$, and can be shown to be equal to $\pi/2 - A(\bar{Y},1)/2 = .3845$ where $A(\cdot)$ is derived in Appendix A (equation (A.1)). The area of the triangle is easily seen to be .7094 and thus we can write the following equations:

$$\begin{aligned} \pi/6 - .3845 &= A + B \\ .7094 &= 3(A + B) + C \end{aligned}$$

which can be solved to yield $A = .0467$, $B = .0923$, and $C = .2921$. We can convert these areas into probabilities by dividing by $\pi/6$ to obtain $P_A = .0891$, $P_B = .1762$, and $P_C = .5579$. Each area is influenced by only a subset of the possible transmitters surrounding it, and we can write the probability that $P$'s transmission is successfully received as:

$$Q = P_A + 2P_B(1-p) + P_C(1-p)^2$$

The fraction of successful traffic then is given by $C(N,p) = g(N)pQ$ which is maximized for $p^* = .4599$ at which point $Q = .4421$. Using this in the expression for $C(N,p)$ allows us to calculate the maximal fraction of successful transmissions in the CSMA network which is given by:

$$C^*(N) = .4504/N$$

Thus we have the efficiency of the CSMA protocol in the multi-hop environment under ideal assumptions is about 48.5%. This performance is in striking contrast to the efficiency of CSMA in the single-hop environment [8] where under commonly held assumptions it has a throughput of about 87%. We may also note that the performance of CSMA in the multi-hop environment under ideal assumptions is not much greater than that of slotted-ALOHA.

## 5 Conclusions

In this paper we have calculated an upper bound for the maximal expected fraction of successful transmissions obtained in a random connected planar packet radio network. The method utilized is notable in that it produced an intuitively pleasing result that $f^*(N) \cong 1/N$ and was simple to calculate. Using this optimal performance as a standard, we compared the slotted-ALOHA and CSMA protocols to it, and have shown that these protocols, for realistic networks, have an efficiency of about 36%, and 48.5% respectively.

## APPENDIX A
### Calculating $A(r_1, r_2)$

In this appendix we will derive an equation for $A(r_1, r_2)$. We will use figure A.1 to facilitate this derivation.

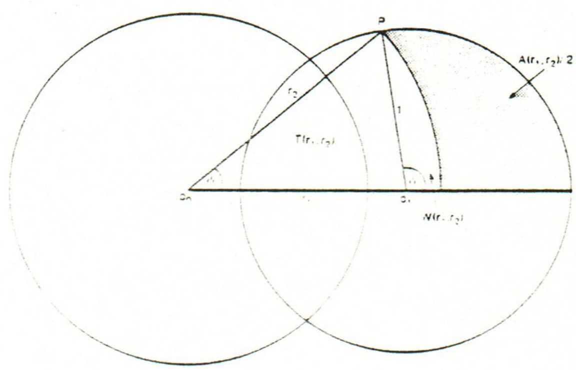


FIGURE A.1
Deriving the Function $A(r_1, r_2)$

In this figure, $T(r_1, r_2)$ is the area formed by the triangle of vertices $p_0$, $p_1$, and $P$, $W(r_1, r_2)$ is the area which added to $T(r_1, r_2)$ would form the sector of radius $r_2$ and angle $\phi$, and $A(r_1, r_2)/2$ is the shaded area in $C_1$. Since $A(r_1, r_2)/2$ is contained in a sector of radius 1 and angle $\psi$, we can write:

$$A(r_1, r_2)/2 = \psi/2 - W(r_1, r_2)$$

but $W(r_1, r_2)$ can easily be seen to be:

$$W(r_1, r_2) = \phi r_2^2/2 - T(r_1, r_2)$$

We can thus derive $A(r_1, r_2)$ if we can derive equations for $\psi$, $\phi$, and $T(r_1, r_2)$. We can use the Law of Cosines repeatedly to obtain:

$$1 = r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi) \quad \rightarrow \quad \phi = \cos^{-1}\left(\frac{r_2^2 + r_1^2 - 1}{2\, r_1 r_2}\right)$$

$$r_2^2 = 1 + r_1^2 - 2r_1 \cos(\pi - \psi) \quad \rightarrow \quad \psi = \cos^{-1}\left(\frac{r_2^2 - 1 - r_1^2}{2\, r_1}\right)$$

We can thus write $T(r_1, r_2)$ in terms of known quantities to get $T(r_1, r_2) = r_1 r_2 \sin(\phi)/2$. Combining this all together we have:

$$A(r_1, r_2) = \cos^{-1}\left(\frac{r_2^2 - 1 - r_1^2}{2\, r_1}\right) - r_2^2 \cos^{-1}\left(\frac{r_2^2 + r_1^2 - 1}{2\, r_1 r_2}\right) + r_1\left(1 - \left(\frac{r_2^2 + r_1^2 - 1}{2\, r_1 r_2}\right)^2\right)^{1/2} \qquad \text{(A.1)}$$

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