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NODAL BLOCKING IN LARGE NETWORKS*

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ABSTRACT

A theoretical study is given for store-and-forward communication networks in which the nodes have finite storage capacity for messages. A node is "blocked" when its storage is filled, otherwise it is "free." A two-state Markov model is proposed for each node, and the number of blocked nodes in the network is shown also to have a two-state Markov process representation. Digital computer simulations substantiate the theoretical results.

INTRODUCTION

Consider a store-and-forward communication network (e.g., see Refs. 1-5) consisting of nodes having finite storage space for messages. During periods of high traffic intensity this storage can be expected to fill from time to time. In this condition the node must refuse incoming messages (which might be accomplished by sending negative acknowledgments) and we then say that the node is "blocked."

As soon as one message is transmitted by a blocked node, it becomes a "free" node. It remains in this state as long as there is at least one empty space in storage that could be used by an arriving message. When the storage fills again, the node re-enters the blocked state.

THE MODEL

Figure 1 shows a simplified model of such a node in the terminology of the ARPA network[1-5]. The Interface Message Processor (IMP), when free, accepts messages into its main storage from two sources: (1) other IMPs like itself, and (2) a HOST which generates and receives messages (as a source and terminal) and communicates with the rest of the network by means of the IMP. Message bits are sent in parallel to the message buffer serving the appropriate output line, as determined by the final destination of the message, and are then transmitted serially to that neighbor. Any of these neighbors can become blocked, thus preventing the use of the output line feeding such neighbors.

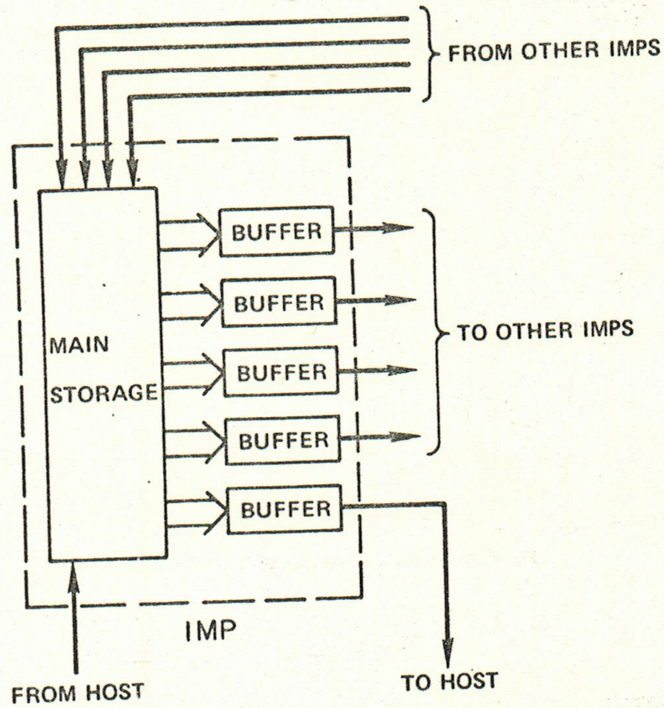

Figure 1. Schematic of a Node

In this paper we study nodal blocking caused by the finite storage room for messages in the IMP and the overutilization of the system. By overutilization, we mean that when the node is accepting messages, its average arrival rate equals or exceeds its average service rate (which is the total output channel capacity divided by the average message length). Elementary queueing theory[6] shows that if (1) the system is underutilized, and (2) there is storage space for approximately twenty messages or more, then under fairly general conditions there will be essentially no blocking.

Nodal blocking is a transient effect which should occur only at peak hours during the day in a well-designed system, but once started it could propagate in both space and time. The analysis of this propagation is difficult for at least three reasons. First, it involves networks of queues, for which only stationary results at best can generally be obtained. Second, the pertinent stochastic processes are dependent, for if a node becomes blocked, it cannot accept messages from its neighbors and their storage will tend to fill at a faster rate. Finally, it is a transient queueing problem and even the simplest of these is very difficult to solve. (For example, the queueing system with Markovian arrivals, a single exponential server, and unlimited waiting room has modified Bessel functions in its time dependent solution[6].)

Since we cannot solve the problem exactly, our goal is to make good approximations that allow us to analyze the system and characterize its blocking behavior in some way. To this end we make the following assumptions:

1. The HOST cannot become blocked (it is an infinite sink)

2.a. Input traffic from the HOST is Poisson

b. Traffic on all lines has the same average rate so that total average traffic into each node is σ messages/sec.

3.a. Message lengths are exponentially distributed

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b. Service (transmission) time on any line is therefore exponentially distributed such that for a node with k blocked neighbors, the rate at which messages exit from that node is $\mu^{(k)}$ messages/sec.

4. Probability of an empty queue in the IMP is approximately zero (since the system is assumed to be overutilized)

ANALYSIS AND RESULTS

Under these assumptions we arrive at a simplified blocking model for a node in the network, and its description as a two-state Markov process is given in Figure 2. If the node is blocked, i.e., in state b,

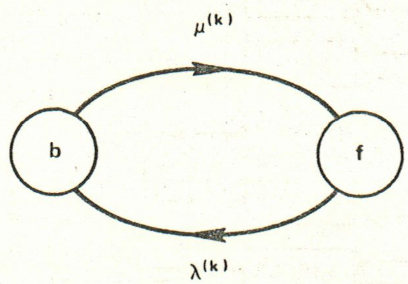


Figure 2. Blocking Model for an Imp

it becomes free in the next instant of time Δt with probability $\mu^{(k)}\Delta t$ where k is the number of blocked neighbors it is experiencing at that time. Similarly, if the node is free, i.e., in state f, it becomes blocked in the next instant of time Δt with probability $\lambda^{(k)}\Delta t$ where k is again the number of blocked neighbors.

Below we show the appropriateness of this model. First, we require the Laplace transform of the interdeparture time probability density $\equiv D(s)$. For any node let $\rho \equiv P_r$[non-empty node] and let the Laplace transform of the probability density of the interarrival time process be $A(s)$. Because we have assumed that the service time is exponential with parameter $\mu^{(k)}$, we know that the Laplace transform of the departure process, conditioned on a non-empty system is $\mu^{(k)}/(s + \mu^{(k)})$. Therefore,

$$D(s) = \frac{\rho\mu^{(k)}}{s + \mu^{(k)}} + (1 - \rho)A(s)\frac{\mu^{(k)}}{s + \mu^{(k)}} \qquad (1)$$

By assumption (4) we have $\rho \approx 1$

$$\therefore D(s) \approx \frac{\mu^{(k)}}{s + \mu^{(k)}} \qquad (2)$$

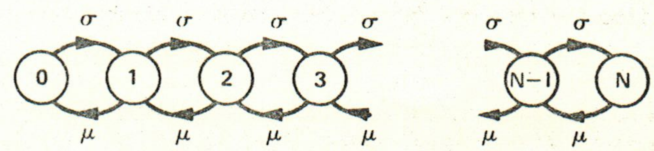
which says that the departure process is a Poisson stream.

We have assumed that the traffic on all lines has the same average rate. If, for example, every node has exactly four neighbors and one HOST, then there are five output lines from each node. All of these lines are equivalent (except that the HOST cannot become blocked) and, by the assumption of exponential message lengths, the departure process from each output line constitutes a Poisson stream when that neighbor is not blocked.

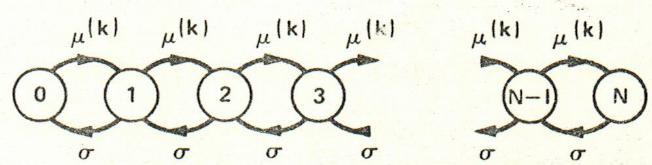
$$\therefore \mu^{(k)} = \frac{5 - k}{5}\mu^{(0)} \qquad k = 0,1, \ldots, 4 \qquad (3)$$

where $\mu^{(0)}$ is a given system parameter and represents the maximum message departure rate from a node. This set of numbers is merely an illustration; any combination can be treated by this model. These results show that we can approximate the time spent in the blocked state as being exponentially distributed with parameter $\mu^{(k)}$.

The time spent in the free state, however, is distributed as the busy period in a queueing system with finite queueing room for customers, as we now show. Consider the state transition diagram or Markov chain model for such a single node queueing system shown in Figure 3a.



a) QUEUE STATE TRANSITIONS



b) DUAL QUEUE STATE TRANSITIONS

Figure 3

The numbers inside the circles represent the number of customers in the node. Customers arrive in a Poisson fashion with parameter σ, and depart after receiving service (exponentially distributed with an average of $1/\mu$ seconds). A busy period begins when a customer arrives to find an empty system (at which time he immediately enters the service facility). Customers arriving during his service time form a queue behind him. With each arrival the system moves to the right along the state transition diagram, because the number in the system is increased by one, and with each service completion, i.e., departure, it moves to the left. The busy period ends the first time the system goes empty after initiation of the busy period.

We now consider a dual queue in which the roles of service and arrival are reversed, and the numbers inside the circles now represent the number of empty places in storage that could be used by arriving messages (see Figure 3b). The free period of the IMP begins with the departure of a message from a previously filled system, i.e., no empty places for arriving messages. With the transmission (departure) the system moves from state 0 to state 1. It continues to move to the right with each transmission and to the left with each arrival. The free period ends the first time the system returns to the 0 state. The correspondence between the primal and dual queues is perfect; thus any results obtained for the busy period in the primal system are applicable to the dual queue free period in the IMP simply by substituting $\mu^{(k)}$ for σ and σ for μ, as in Figs 3a, b.

The busy period for a finite queueing room system is difficult to obtain, but the result for unlimited queueing room is well known. The probability density of the length t of the busy period in such a system is

$$p(t) = \frac{1}{t\sqrt{\rho}} e^{-(\sigma+\mu)t} I_1(2t\sqrt{\sigma\mu}) \tag{4}$$

where ρ, the utilization factor $= (\sigma/\mu) < 1$ and $I_1(x)$ is the modified Bessel function of the first kind, of order one[7]. If the size of the queueing room is greater than 20, the solution for unlimited queueing room is a good approximate solution to the limited queueing room problem. (This follows since we have assumed $P[\text{empty IMP}] \approx 0$; but the $P[\text{empty IMP}]$ corresponds to the probability of being in state N (i.e., all N spaces are empty) in Figure 3b, and thus an increase in N will not seriously affect our results.) Since we have assumed overutilization, we have $(\mu^{(0)}/\sigma) \leq 1$, and we are justified in substituting this (or $\mu^{(k)}/\sigma$) for ρ. Thus we get the following for the probability density of the length t of the time spent in the free state:

$$p(t) = \frac{1}{t}\sqrt{\frac{\sigma}{\mu^{(k)}}}\, e^{-(\sigma+\mu^{(k)})t} I_1(2t\sqrt{\sigma\mu^{(k)}}) \tag{5}$$

As the ratio $\mu^{(k)}/\sigma$ approaches 0, i.e., as the system becomes more overutilized, this density approaches that of the exponential distribution. To arrive at a more tractable model, we approximate the free period distribution by the exponential distribution having the same mean value. The mean value of the busy period in the original system is easy to obtain, and is given by $1/\mu(1-\rho)$. Therefore, as an approximation to the free period in the IMP, we take an exponential distribution with mean value $1/(\sigma-\mu^{(k)})$, i.e., with a parameter,

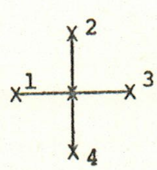
$$\lambda^{(k)} = \sigma - \mu^{(k)} \tag{6}$$

For the marginal case, $\sigma = \mu^{(0)}$, elementary queueing theory[6] shows that we must take

$$\lambda^{(0)} = \frac{\sigma}{N} \qquad \text{for } \sigma = \mu^{(0)} \tag{7}$$

where N is the size of the storage capacity in the IMP.

Our model for the blocking IMP is thus a two-state Markov process or, in the language of renewal theory, an alternating Poisson renewal process[8]. One way to describe the dynamics of a network of such nodes is to examine the probability that any given node is blocked at some time t. Consider a node with four neighbors numbered 1 to 4:



Let $P^k(t) = P[k \text{ neighbors blocked at time } t]$ (8)

and let $p(t) = P[\text{node blocked at time } t]$ (9)

Then, from elementary considerations, we have (correct to within $o(\Delta t)$)

$$p(t+\Delta t) = (1-p(t))\sum_{k=0}^{4} P^k(t)\lambda^{(k)}\Delta t + p(t)\Big(1-\sum_{k=0}^{4} P^k(t)\mu^{(k)}\Delta t\Big)$$

where from Eq. (3) $\mu^{(k)} = \mu^{(0)} - (k/5)\mu^{(0)}$

and from Eq. (6) $\lambda^{(k)} = \sigma - \mu^{(k)} = \sigma - \mu^{(0)} + (k/5)\mu^{(0)}$

for $\sigma > \mu^{(0)}$.

We also note that

$$\lambda^{(k)} + \mu^{(k)} = \sigma \tag{10}$$

$$\text{Thus,}\quad \frac{p(t+\Delta t) - p(t)}{\Delta t} = (1-p(t))\sum_{k=0}^{4} P^k(t)\lambda^{(k)} - p(t)\sum_{k=0}^{4} P^k(t)\mu^{(k)}$$

Letting Δt approach 0, we have

$$\frac{dp(t)}{dt} = -p(t)\sum_{k=0}^{4} P^k(t)(\lambda^{(k)} + \mu^{(k)}) + \sum_{k=0}^{4} P^k(t)\lambda^{(k)}$$

$$= -\sigma p(t)\sum_{k=0}^{4} P^k(t) + \sum_{k=0}^{4} P^k(t)\Big(\sigma - \mu^{(0)} + \frac{k}{5}\mu^{(0)}\Big)$$

$$= -\sigma p(t) + \sigma - \mu^{(0)} + \frac{\mu^{(0)}}{5}\sum_{k=0}^{4} kP^k(t) \tag{11}$$

This can be simplified by noting that

$$E[\text{number of blocked neighbors at time } t] = \sum_{k=0}^{4} kP^k(t) \tag{12}$$

where E denotes expectation. Define the indicator function.

$$f_n(t) = \begin{cases} 1 & \text{if node } n \text{ is blocked at time } t \\ 0 & \text{otherwise} \end{cases}$$

Now let

$$p_n(t) = P[\text{node } n \text{ is blocked at time } t]$$

then

$$E[f_n(t)] = p_n(t) \tag{13}$$

Further, from Eq. (12), we have that

$$\sum_{k=0}^{4} kP^k(t) = E\Big(\sum_{n\in N} f_n(t)\Big)$$

$$= \sum_{n\in N} E(f_n(t)) \tag{14}$$

where N is the set of neighbors for this node (which we number 1,2,3,4). From Eqs. (13) and (14) we get

$$\sum_{k=0}^{4} kP^k(t) = p_1(t) + p_2(t) + p_3(t) + p_4(t) \tag{15}$$

Finally, from Eqs. (11) and (15) we have the result

$$\frac{dp(t)}{dt} = -\sigma p(t) + \sigma - \mu^{(0)} + \frac{\mu^{(0)}}{5}(p_1(t) + p_2(t) + p_3(t) + p_4(t)) \tag{16}$$

This relation can also be derived from epidemiology by considering nodal blocking as a deterministic epidemic without migration and with but two kinds of individuals, infected and susceptible[9].

Adjacent nodes have nearly equal probabilities of being blocked. Consider the case when all of these probabilities are exactly equal (as an approximation). Then from Eq. (16)

$$\frac{dp(t)}{dt} = -\sigma p(t) + \sigma - \mu^{(0)} + \frac{4}{5}\mu^{(0)}p(t)$$

$$= -(\sigma - \frac{4}{5}\mu^{(0)})p(t) + \sigma - \mu^{(0)}$$

which has the solution

$$p(t) = \left[p(0) - \frac{\sigma - \mu^{(0)}}{\sigma - \frac{4}{5}\mu^{(0)}}\right] e^{-(\sigma - \frac{4}{5}\mu^{(0)})t} + \frac{\sigma - \mu^{(0)}}{\sigma - \frac{4}{5}\mu^{(0)}} \tag{17}$$

Now consider the alternating Poisson renewal process shown in Figure 4. There are two states, called (B) and

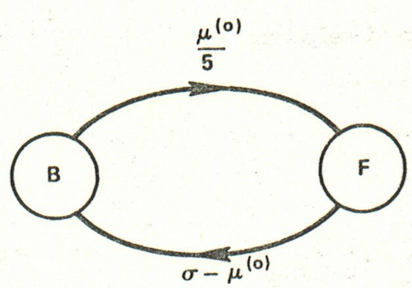


Figure 4. Network Model

free (F). If the system is in the blocked state (B) at time t, it goes free (state F) in the next instant Δt with probability $(\mu^{(0)}/5)\Delta t$. In similar fashion, the probability that it leaves the free state and re-enters the blocked state is $(\sigma - \mu^{(0)})\Delta t$. Therefore, the probability that it is in the blocked state at time $t + \Delta t$ is

$$p_B(t + \Delta t) = p_B(t)(1 - \frac{\mu^{(0)}}{5}\Delta t) + (1 - p_B(t))(\sigma - \mu^{(0)})\Delta t$$

$$\therefore \frac{dp_B(t)}{dt} = -p_B(t)(\sigma - \frac{4}{5}\mu^{(0)}) + (\sigma - \mu^{(0)})$$

or

$$p_B(t) = \left[p_B(0) - \frac{\sigma - \mu^{(0)}}{\sigma - \frac{4}{5}\mu^{(0)}}\right] e^{-(\sigma - \frac{4}{5}\mu^{(0)})t} + \frac{\sigma - \mu^{(0)}}{\sigma - \frac{4}{5}\mu^{(0)}} \tag{18}$$

This is the same as Eq. (17) which was obtained for the probability that a node is blocked at time t! In a large homogeneous system the fraction of blocked nodes may be closely approximated by the probability that any one of them is blocked. Therefore, the fraction of blocked nodes at time t in a large uniformly connected (i.e., two-dimensional lattice) network is approximately equal to the probability that the two-state Markov process shown in Figure 4 is in the blocked state at time t. Thus we may take this two-state Markov process as a model for the network.

So far we have presented only aggregate results. To obtain the probability that any given node in the network is blocked at time t we must consider a system of equations of the form (See Eq. (16))

$$\frac{dp_i(t)}{dt} = -\sigma p_i(t) + \sigma - \mu^{(0)} + \frac{\mu^{(0)}}{5}(p_j(t) + p_k(t) + p_\ell(t) + p_m(t))$$

for each node i in the network with neighbors j, k, ℓ, and m. These equations are obviously of the form

$$\dot{P}(t) = AP(t) + C \tag{19}$$

If there are M nodes in the net, then $P(t)$ is the $M \times 1$ matrix whose i^{th} component is the probability that node i is blocked at time t. A is an $M \times M$ constant matrix and C is an $M \times 1$ constant matrix. The solution is well known:

$$P(t) = e^{At}P(0) + A^{-1}(e^{At} - I)C \tag{20}$$

For a small net this solution poses no difficulty, but for a large one the required matrix computations rapidly get out of hand. There are some special cases which are solvable, however, and we obtain the solution for one of these below.

Consider a network consisting of 1024 nodes arranged in a 32 x 32 (n x n) grid. For this system the matrix A is $n^2 \times n^2$ or 1024 x 1024 and takes the following form:

$$A = \begin{bmatrix} D & \Lambda & & & & O \\ \Lambda & D & \Lambda & & & \\ & \Lambda & D & \Lambda & & \\ & & & \cdots & & \\ & & & \Lambda & D & \Lambda \\ O & & & & \Lambda & D \end{bmatrix} \tag{21}$$

$$\text{where } D = \begin{bmatrix} a & b & & & & \\ b & a & b & & 0 & \\ & b & a & b & & \\ & & & \cdots & & \\ & 0 & & b & a & b \\ & & & & b & a \end{bmatrix}_{n \times n} \qquad (22)$$

and
$$\Lambda = bI_n \qquad (23)$$

where $a = -\sigma$, $b = \frac{\mu^{(0)}}{5}$, and I_n is the

$n \times n$ identity matrix (25)

This observation holds for a square grid with any number of nodes n on a side. (See the Appendix for the complete solution for $P(t)$ for arbitrary n.) This case of $n = 32$ was simulated and is described in the following section.

SIMULATION RESULTS

Simulation of a network of 1024 nodes employing the Markovian inter-event time assumption substantiates the approximations described in the theoretical results above. Two different simulation programs have been run on the UCLA XDS Sigma-7 computer.* The first was for a network arranged in a square grid 32 x 32. Each node is connected to its four nearest neighbors (a lattice) except in the case of nodes along the border which have only three nearest neighbors (or two nearest neighbors in the case of the four corner nodes). When a node changes state, new event times are chosen for it and for all of its nearest neighbors based on the new number of blocked neighbors. The memoryless property of the exponential distribution simplifies the calculations.

The second program simulated a randomly connected graph in which each node was given exactly four neighbors.

Comparison of the two-state Markov process model and the simulation results for the lattice and the random graph are shown in Figure 5 for one set of parameters σ and $\mu^{(0)}$ starting from a net that is completely blocked. Figure 6 shows the results when the network begins with all of its nodes in the free state. In Figure 7 results are compared for the model and the two-dimensional integer lattice in which each node is assumed to have eight neighbors. This was accomplished by extending the nearest neighbor definition to include nodes which are diagonally adjacent. The results in Eq. (18) are extended in the obvious way. Figure 8 compares simulation results on the lattice of degree four, when a free node with k blocked neighbors is considered k-fourths blocked, to the predicted trajectory based on a non-linear "partial blocking" model. The agreement with the simulations is generally good, and the model is sufficiently general to treat a variety of cases.

CONCLUSIONS

Two new models that may have application to store-and-forward communication networks are presented in this paper. The probabilistic model for nodal blocking due to finite storage space is shown in Figure 2. The second model, and the main result of this work, is that the fraction of blocked nodes in a network of such nodes has

*During simulation the net activity was displayed on a Digital Equipment Corporation 340 Precision Display CRT.

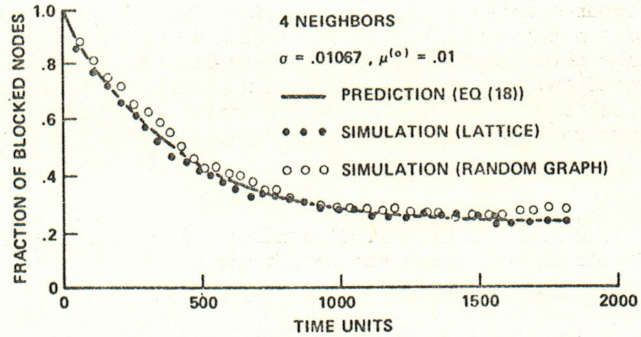


Figure 5. Fraction of Blocked Nodes I

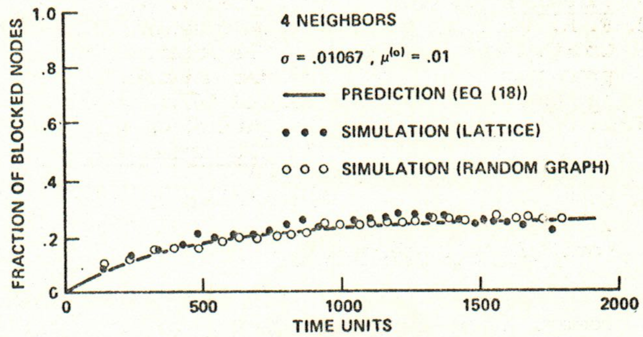


Figure 6. Fraction of Blocked Nodes II

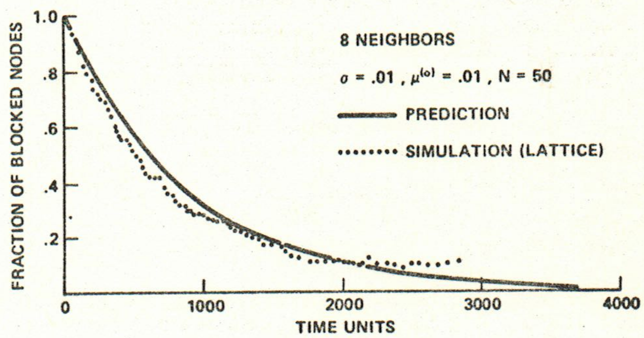


Figure 7. Fraction of Blocked Nodes III

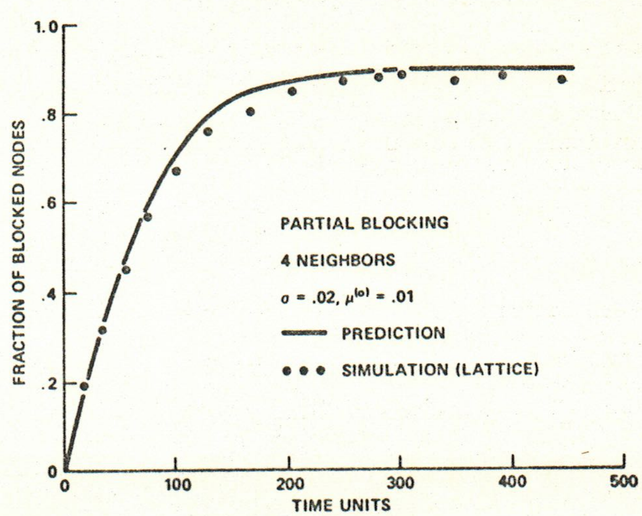


Figure 8. Fraction of Blocked Nodes IV

a two-state Markov process representation (Figure 4 and Eq. (18)). Figures 5-8 verify that the network model compares well with results obtained from the simulation of a network of two-state Markovian nodes in which the time spent in either state is a function only of the state and the number of blocked neighbors. Finally, the model is sufficiently general to treat a variety of network configurations and parameters.

ACKNOWLEDGMENTS

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APPENDIX

We must first find the eigenvalues γ_v of D which are the solutions of $|D - \gamma I| = 0$. Let a stand for $a - \gamma$ in D; we wish to find the zeros of the determinant of D. Expanding by the elements of the top row, we note the following recurrence relation for the determinant Δ_n of the $n \times n$ matrix D:

$$\Delta_n = a\Delta_{n-1} - b^2\Delta_{n-2}$$

with initial conditions $\Delta_1 = a$, $\Delta_0 = 1$, $\Delta_{-1} = 0$. Following Grenander and Szego[10] we substitute $a = 2b \cos \theta$, assume a solution of the form $\Delta_n = \rho^n$, and solve the resulting quadratic in ρ. After satisfying the initial conditions the result is simply

$$\Delta_n = b^n \frac{\sin(n+1)\theta}{\sin \theta}$$

which vanishes for $\theta = v\pi/n{+}1 \qquad v = 1,2, \ldots, n$

Therefore, the eigenvalues of D are

$$a - 2b \cos \frac{v\pi}{n+1} \qquad v = 1,2, \ldots, n$$

which are all distinct. The eigenvectors are the solutions of

$$\begin{bmatrix} a & b & & & \bigcirc \\ b & a & b & & \\ & b & a & b & \\ & & & \cdots & \\ \bigcirc & & & b & a \end{bmatrix} \begin{bmatrix} X_{v1} \\ X_{v2} \\ X_{v3} \\ \cdots \\ X_{vn} \end{bmatrix} = \gamma_v \begin{bmatrix} X_{v1} \\ X_{v2} \\ X_{v3} \\ \cdots \\ X_{vn} \end{bmatrix}$$

It is easy to verify that the normalized solutions are

$$X_{vk} = \frac{(-1)^{n-k}}{\sqrt{\frac{n+1}{2}}} \sin \frac{kv\pi}{n+1}$$

so that the (i,j) element of e^D is

$$e^D_{i,j} = \sum_{v=1}^{n} e^{\gamma_v} X_{vi} X_{vj}$$

and

$$D^{-1}_{i,j} = \sum_{v=1}^{n} (\gamma_v)^{-1} X_{vi} X_{vj}$$

where

$$\gamma_v = a - 2b \cos \frac{v\pi}{n+1} \quad \text{and} \quad X_{vk} = \frac{(-1)^{n-k}}{\sqrt{\frac{n+1}{2}}} \sin \frac{kv\pi}{n+1}$$

Similarly, it is easy to show that the transformation R^*AR (where R^* is the transpose of R) where

$$R \equiv \begin{bmatrix} X_{11}I_n & \cdots & X_{v1}I_n & \cdots & X_{n1}I_n \\ X_{12}I_n & \cdots & X_{v2}I_n & \cdots & X_{n2}I_n \\ & & & & \\ X_{1n}I_n & \cdots & X_{vn}I_n & \cdots & X_{nn}I_n \end{bmatrix} \text{ with } X_{vk} \text{ as given}$$

above reduces A to the quasi-diagonal form

$$\begin{bmatrix} M_1 & & \bigcirc \\ & M_2 & \\ & & \cdots \\ \bigcirc & & M_n \end{bmatrix}$$

where $\qquad M_v = D - 2b \cos \frac{v\pi}{n+1} I_n$

Since M_v is equal to D with a change of the diagonal element, we have that the (k,l) element of the (i,j) block of e^A is

$$e^A_{i,j;k,l} = \sum_{v=1}^{n} X_{vi}X_{vj} \sum_{p=1}^{n} \exp(a-2b \cos\frac{v\pi}{n+1} -2b \cos\frac{p\pi}{n+1}) X_{pk}X_{pl}$$

and

$$A^{-1}_{i,j;k,l} = \sum_{n=1}^{n} X_{vi}X_{vj} \sum_{p=1}^{n} (a-2b \cos\frac{v\pi}{n+1} -2b \cos\frac{p\pi}{n+1})^{-1} X_{pk}X_{pl}$$

where

$$X_{vk} = \frac{(-1)^{n-k}}{\sqrt{\frac{n+1}{2}}} \sin \frac{kv\pi}{n+1}$$

In our system $a = -\sigma$ and $b = \mu^{(0)}/5$ so the time constants, i.e., the arguments in each of the exponentials appearing in the solution for e^{At} are of the form

$$-\sigma t - \frac{2\mu^{(0)}}{5} t \left[\cos \frac{v_i \pi}{n+1} + \cos \frac{v_j \pi}{n+1} \right]$$

which takes on its smallest absolute value for $v_i = v_j = n$. Thus the motion of the system is bounded by

$$\exp - (\sigma - \frac{4}{5} \mu^{(0)} \cos \frac{\pi}{n+1}) t$$

The number n is the square root of the number of nodes in the square lattice. This result shows that as $n \to \infty$ the system attains its steady state at a rate

$$\exp - (\sigma - \frac{4}{5} \mu^{(0)}) t$$

which agrees with simulation results for $n = 32$.