QUEUEING ANALYSIS OF THE ORDERING ISSUE IN A DISTRIBUTED DATABASE CONCURRENCY CONTROL MECHANISM*

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Abstract

Preserving database consistency is a major task in a distributed environment where sites may contain portions or complete replicas of databases and where no centralized control is allowed for reliability reasons. This paper presents some preliminary results from a queueing theoretic analysis based on Le Lann's ticketing algorithm. In this algorithm, as in others, database requests can be received at a site in an order different from that in which they were generated due to variable communication delays. One method to ensure mutual consistency between two or more copies of the database is to require that at each site requests are processed in the order in which they originated and not in the (arbitrary) order in which they arrive at the database copy. We concentrate in this paper on the characteristics of the disorder introduced in the database request sequence by the network delays.

The results presented here are expected to be useful in a total system performance analysis.

I. Introduction

The development of efficient, reliable computer communication networks (both public and private) has led naturally to the problem of the distribution of resources among the hosts attached to the network. In particular, the distribution of data has triggered a great many theoretical and experimental investigations. In addition to reliability, fault-tolerance, responsiveness and other network objectives, distributed database systems are subject to severe integrity consistency constraints. Preserving the database consistency becomes a major task in a distributed environment where sites may contain portions or complete replicas of databases and where no centralized control is allowed for reliability reasons. Basically, consistency refers to two constraints: *internal consistency* and *mutual consistency*. The first is fundamental to database systems, whether distributed or not, and is related to semantic requirements of the database. The second is inherent to the distributed environment where the variable nature of intercommunication delays between sites may interfere with the correct order of update operations. Briefly speaking, mutual consistency means that all replicated portions of a database must converge to identical copies should update activity be interrupted.

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A number of important concurrency control schemes have been developed in order to preserve the mutual consistency of distributed database systems, and a comprehensive discussion of proposed concurrency control algorithms can be found in a recent report by Bernstein and Goodman [1]. As the number of schemes grows it becomes imperative that some analytical quantitative tools become available to help identify performance characteristics of the various schemes. Work by Lee [5] dealt with a comparison of the network semaphore scheme, the hopping permit scheme and the adaptive hopping scheme. Garcia-Molina [2] developed queueing models for some centralized schemes which he compared with some decentralized voting schemes. Gelenbe and Sevcik [3] defined the notions of coherence and promptness for redundant (multiple copies) distributed databases. They developed a queueing model that permits the analytical evaluation of these quantities. Their model relies on the use of perfect time synchronization through the network, and provides an interesting approach to capture the disparity of information at a certain time between real copies and an ideal instantaneously updated copy. Such dispersion is in fact due to variable network delays.

In this paper our main focus will be to identify the effect of network delays on the order of updates. Our ultimate goal is an analysis of the "ticketing mechanism" developed by Le Lann, a version of which is being implemented in the Sirius-Delta Project at INRIA (Institut National de Recherche en Informatique et Automatique, France) [6,7]. While our results here apply to other concurrency control algorithms, we will assume Le Lann's ticketing mechanism for the sake of a concrete example. Briefly speaking, we consider a fully redundant multiple copy distributed database system. Updates to the data base generated from independent sites are allocated sequentially increasing ticket numbers before entering the network. A circulating token is in charge of delivering tickets. On the data management side, at each site updates must be processed according to the order of their associated tickets so that mutual consistency is preserved. In another version of this concurrency control scheme updates are processed in a temporary workfile in the order of delivery from the network but they become final only when conflicts are no longer possible, i.e., when all lower order (ticketed) updates have also been processed.

In this paper, updates from all sites are modelled as a Poisson stream of customers to an infinite server system that plays the role of the network. Special *star* customers are identified as those that leave the network "in order" with regards to lower ticketed updates. These customers can then be safely served by the data base manager (according to the initial version of the algorithm) and mutual consistency is maintained. Their end of transmission through the network will make all *consecutive* higher order updates that previously

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departed the network eligible for service. The output process of the star customers, as well as the distribution of the number of customers which become eligible with the star customer, will be characterized under an exponential network service time assumption. The distribution of the waiting time of out-of-order customers to become eligible for service by the database manager is also provided as well as other interesting analytic properties. Numerical results are also given which exhibit the behavior of this loss-of-order phenomenon that is inherent to networks.

II. The Model

In this section the main features of Le Lann's ticketing algorithm are first presented. In particular, functions that pertain to the ordering of updates (i.e., assignment of increasing numbers) and the actual utilization of that order at the data management end are reviewed. Reliability features will not be considered in this paper. Second, a queueing model is developed to basically capture the effect of order, or more correctly, of "disorder". The concern here is not to model the complete system which is, however, the ultimate goal of the research.

II.1 Basic Features of Le Lann's Ticketing Algorithm

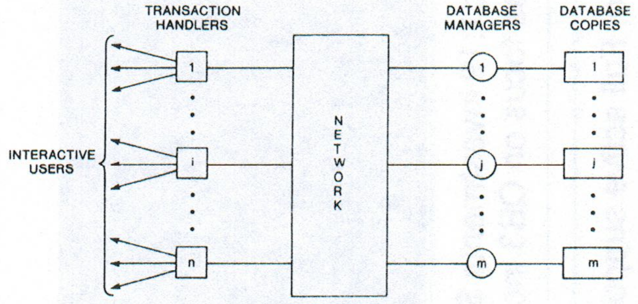

Fig. 1. Fully redundant distributed database system.

The system considered and shown in Fig. 1 is composed of dispersed interactive users that forward their requests to local *transaction handlers.* Transaction handlers interpret and execute user's requests that may involve reading, updating, deleting, and creating data items in the database. Full copies of the database are stored and controlled by database managers located at storage sites. Communication between sites is handled by a data communication network. We are concerned here with updating requests where the order of execution by the database managers is crucial in order to preserve mutual consistency. (Other types of requests will be ignored in the analysis below.) Internal consistency is taken care of through locking mechanisms; however this may lead to deadlock situations which can also be prevented through the preservation of the order of execution.

The mechanism used is called the *circulating sequencer.* Transaction Handlers are *conceptually* organized in a virtual ring. A particular message called *the token* circulates in the ring. The token delivers sequential and unique integer values called *tickets.* Upon reception of the token a transaction handler will acquire tickets for its pending operations on the database (note that tickets serve to time stamp operations on the database). The sequence number is incremented and the token is passed on to the successor transaction handler on the virtual ring. Since we consider a fully redundant database and only update operations, all such operations must be performed on all copies of the database. Two basic strategies are possible by the database manager: either **i)** perform operations only according to increasing sequential ticket numbers or **ii)** perform the operations as they arrive on temporary space and roll back in case a later operation (i.e., a smaller ticket number) would conflict with some already executed operations.

Other details on the breakdown of operations into smaller indivisible units and on "commit" protocols are omitted here.

It is now clear that the order in which operations reach the destination database managers is of extreme importance in the study of the behavior of this ticketing algorithm. In what follows we develop a queueing model of that process which is based on the first, but probably more conservative, strategy. Extensions of the model to include the second strategy are currently under investigation.

II.2 The Infinite Server Model

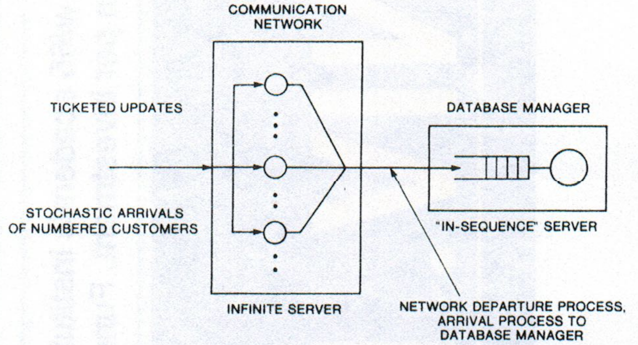

Fig. 2. The infinite server queueing model.

In order to study the basic behavior of the out-of-order problem we model the communication network as a single queueing system with an *infinite number of servers* (see Fig. 2). The generation of updates and their numbering by the external transaction handlers is modelled as a stochastic process to the infinite-server system. The numbering corresponds to the assignment of increasing successive numbers to consecutive arrivals. The first "update" (i.e., the first arrival) will be assigned "ticket number 1", the second, 2 and so on.

An update will be referred to as a *customer* in the balance of this paper. The customer with sequence number n will be denoted by C_n. The service times of customers (i.e., the time an update spends in the network) are random variables. Due to the random nature of network delay, a customer, say C_n, may leave the network before some other customers holding lower ticket numbers. However, as mentioned above, the receiving database manager only serves customers in increasing sequential order. In this case C_n will be referred to as an *out-of-sequence* (or interchangeably out-of-order) customer. Let d_n be the departure time from the network for C_n (i.e., it is the arrival time to a database manager). We can now state the following definitions:

Out-of-Sequence Customer

C_n is an out-of-sequence customer iff $\exists\, k<n$ s.t. $d_k > d_n$

In-Sequence Customer or Star Customer

C_n is a star customer iff $\forall k < n \;\; d_k < d_n$

In other words C_n is a star customer if he departs the network after all lower ticketed customers. As an example consider the following initial input sequence:

1 2 3 4 5 6 7 8 9 10 11 12

A possible output sequence may be:

4 2 3 1 6 5 7 10 9 12 8 11

The numbers reflect, from left to right, the increasing value of network departure times. C_4 is the first out and C_{11} the last out. Note that 1, 5, 7, 8, 11 are star customers whereas all others are out-of-sequence customers. The database manager, being an *in-sequence server,* will initially receive 4, 2, 3, but only when 1 arrives will service begin. Hence the database manager is not work-conservative in the sense that work may be present while the server is refusing to do it. However, a free server never refuses work to an arriving star customer, so that after the star customer is served some of the waiting out-of-sequence customers may become eligible for service. Let us define eligibility more formally.

Eligibility: A waiting customer, say C_n, at the database management station is eligible for service if for all k such that $k<n$ C_k has been either served by the database manager or is waiting to be served. Consequently if C_n is an out-of-sequence customer which arrives at the database management station then he must wait until after the arrival of all customers with lower ticket numbers. It is obvious that C_n will become eligible upon the arrival of a star customer. However it is also true that the arrival of a star customer does not necessarily provoke the transition of all non-eligible customers waiting at the database station to the eligible state. In the previous example:

Upon arrival of:	Customers that become eligible:
1	2, 3, 4
5	6
7	–
8	9, 10
11	12

Note that when C_8 arrives, only 9 and 10 become eligible whereas 12 becomes eligible only when 11 arrives.

As far as the service requirements are concerned, the database manager sees the arriving process of customers as a "bulk arrival" process where the head of a bulk is a star customer and the rest of the bulk is composed of the customers that become eligible upon arrival of the star customer. By definition:

In-Sequence Bulk (ISB): The ISB of a star customer C_n is composed of the sequence of *consecutive* higher ticketed customers that depart the network before C_n (i.e., they arrive at the database management station before C_n). The ISB of a star customer C_n is exactly of size k iff $C_{n+1}, C_{n+2}, \ldots, C_{n+k}$ depart the network before C_n and C_{n+k+1} departs after C_n. (It is immediately apparent that we can similarly define the ISB for an arbitrary customer and in Section III we will have occasion to use this notion. However, unless explicitly stated, ISB will be used only in connection with star customers.)

Bulk: A bulk is composed of an ISB and the corresponding star customer. Let us denote by S_m the m^{th} star customer; then in the previous example we have:

	STAR CUSTS.	ISB	ISB SIZE	BULK	BULK SIZE
S_1	C_1	C_2, C_3, C_4	3	C_1, C_2, C_3, C_4	4
S_2	C_5	C_6	1	C_5, C_6	2
S_3	C_7	–	0	C_7	1
S_4	C_8	C_9, C_{10}	2	C_8, C_9, C_{10}	3
S_5	C_{11}	C_{12}	1	C_{11}, C_{12}	2

In order to analyze the queueing system at the data base station we need to characterize the following:

i. The network departure process of the star customers from the network.

ii. The distribution of the ISB size of a star customer.

iii. The time an out-of-sequence customer spends at the database management station before it becomes eligible.

Let us note that the time spent waiting for eligibility is only one component of the total waiting time of a customer at the database management station. In fact we may view this component as the waiting time in a "virtual bulk collection box" as shown in Fig. 3. This time will be referred to as the *eligibility waiting time.* Note that the size of the ISB of a star customer provides a quantitative measure of the magnitude of the network disorder, whereas the virtual waiting time provides a measure of the duration of that disorder. Next we proceed with the analysis.

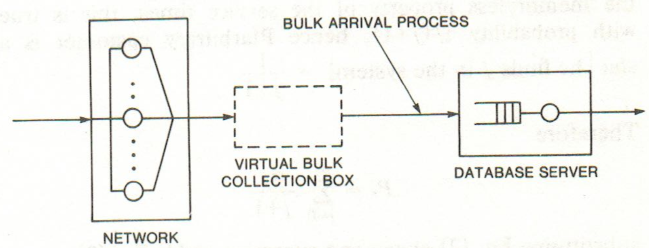

Fig. 3. Virtual bulk collection box.

III. Analysis

First we introduce some assumptions with regard to the distributions of the interarrival times and service times of customers in the network. In this paper, we use the Markovian assumptions for both distributions. Some more general results will be presented in another report.

The interarrival times and service times of customers to the network are independent and exponentially distributed random variables with parameters λ and μ respectively. Denoting the interarrival time density function by $a(t)$ and the network service time density function by $f(x)$

$$a(t) = \lambda e^{-\lambda t} \qquad t \geqslant 0$$

$$f(x) = \mu e^{-\mu x} \qquad x \geqslant 0 \tag{1}$$

In other words the communication network is modelled as an M/M/∞ queueing system. For an M/M/∞ in steady state the probability of having j customers in the system is equal to the probability that an arrival finds j in the system and is given by:

$$\alpha_j = \frac{\rho^j}{j!} e^{-\rho} \tag{2}$$

where $\rho = \lambda / \mu$.

III.1 Probability of a Star Customer: P_*

Theorem 1: The steady state probability that an arbitrary customer is a star customer is given by

$$P_* = \frac{1-e^{-\rho}}{\rho} \tag{3}$$

Also the average bulk size is

$$\overline{G} = \frac{1}{P_*} = \frac{\rho}{1-e^{-\rho}} \tag{4}$$

Moreover the probability density function (p.d.f.) of the (network) service time of a star customer is:

$$f_s(x) = \frac{1}{P_*} \mu e^{-\mu x} e^{-\rho e^{-\mu x}} \tag{5}$$

Proof: Recall that a customer is a star customer only if it departs the net after all lower numbered customers. Consequently, the only customers to consider are those who are still in service (in the network) at the arrival time of our arbitrary tagged customer. If we condition on the fact that there are k customers at the arrival time, in order for the tagged customer to be a star its service time must be greater than the residual service times of all j customers. Due to the memoryless property of the service times, this is true with probability $1/(j+1)$, hence P[arbitrary customer is a star | he finds j in the system] $= \frac{1}{j+1}$

Therefore

$$P_* = \sum_{j=0}^{\infty} \frac{\alpha_j}{j+1}$$

substituting Eq. (2) above and summing yields Eq. (3).

Obviously, to each star customer corresponds a bulk with regard to the arrival process to the database system. Hence the departure rate of bulks is

$$\lambda_* = \lambda P_* \quad \text{also} \quad \lambda = \lambda_* \overline{G}$$

which proves Eq. (4).

Let $F(s,x)$ be the joint probability distribution function that an arbitrary customer is a star customer and has a service time, $\underset{\sim}{x}$, less than or equal to x and let $f(s,x)$ be the corresponding p.d.f.; thus

$$f(s,x)dx = P[\text{arbitrary customer is a star, } x < \underset{\sim}{x} \leqslant x+dx]$$

If we condition on the fact that the customer finds j in the system, then

$$f(s,x) = \sum_{j=0}^{\infty} f(s,x \mid j)\alpha_j \tag{6}$$

But,

$$f(s,x \mid j)dx = P[\text{customer is a star, } x < \underset{\sim}{x} \leqslant x+dx$$

$$\mid \text{customer finds j customers in system}]$$

$$= P[x < \underset{\sim}{x} \leqslant x+dx \mid \text{customer is a star, customer finds j}]$$

$$\cdot P[\text{customer is a star} \mid \text{customer finds j}]$$

If $x_1, x_2, \cdots, x_j$ are the service times of the j customers in service upon arrival, then the first term of the product is equal to

$$P[x < \underset{\sim}{x} \leqslant x+dx \mid \underset{\sim}{x} \geqslant x_i \quad i=1,\cdots,j]$$

$$= P[x < \underset{\sim}{x} \leqslant x+dx \mid \underset{\sim}{x} \geqslant \max_i\{x_i\}]$$

$$= \frac{P[\max\{x_i\} \leqslant \underset{\sim}{x} \mid x < \underset{\sim}{x} \leqslant x+dx]\ P[x < \underset{\sim}{x} \leqslant x+dx]}{P[\underset{\sim}{x} \geqslant \max_i\{x_i\}]}$$

The x_i's $i=1,\cdots,j$ and $\underset{\sim}{x}$ are i.i.d.'s and exponentially distributed and thus

$$P[\max\{x_i\} \leqslant x] = [F(x)]^j$$

Note also as mentioned above that

$$P[\underset{\sim}{x} \geqslant \max_{i=1,\cdots,j}\{x_i\}]$$

$$= P[\text{customer is a star} \mid \text{customer finds j}]$$

$$= \frac{1}{j+1}$$

Consequently

$$f(s,x \mid j) = [F(x)]^j f(x) \tag{7}$$

Substituting Eqs. (2) and (7) into (6) we find

$$f(s,x) = f(x)e^{-\rho(1-F(x))} = \mu e^{-\mu x} e^{-\rho e^{-\mu x}} \tag{8}$$

Finally

$$f_s(x)dx = P[x < \underset{\sim}{x} \leqslant x+dx \mid \text{customer is a star}]$$

which is also equal to

$$f_s(x)dx = \frac{P[x < \underset{\sim}{x} \leqslant x+dx, \text{ customer is a star}]}{P[\text{customer is a star}]}$$

hence

$$f_s(x) = \frac{f(s,x)}{P_*} \tag{9}$$

□

III.2 Distribution of Bulk Size

Before we derive the distribution of the ISB size of a star customer, we prove the following lemma.

Lemma 1: In steady state, the joint probability density function of the ISB size of an *arbitrary* customer and the interarrival time to the network between this customer and the first corresponding out-of-sequence customer given x, the service time of the arbitrary customer, is:

$$h_k(t\mid x) = \begin{cases} \dfrac{\lambda^k}{k!}\lambda e^{-\lambda t}\left[\displaystyle\int_0^t F(x-u)\,du\right]^k (1-F(x-t)) & t<x \\ \dfrac{\lambda^k}{k!}\lambda e^{-\lambda t}\left[\displaystyle\int_0^x F(x-u)\,du\right]^k & t\geqslant x \end{cases} \tag{10}$$

Also the marginal distributions of respectively the bulk size and the interarrival times are:

$$h_k(\cdot\mid x) = \int_0^\infty h_k(t\mid x)\,dt \tag{11}$$

and

$$h(t\mid x) = \sum_{k=0}^{\infty} h_k(t\mid x) \tag{12}$$

Proof: Let C_n be an arbitrary tagged (selected) customer denoted by TC, whose network service time is known and equal to x (see Fig. 4). As defined earlier, the ISB of C_n, even though C_n is not necessarily a star customer, is of size k iff $C_{n+1}, C_{n+2}, \cdots, C_{n+k}$ leave the network before C_n and C_{n+k+1} leaves the network after C_n. C_{n+k+1} is, therefore, the first corresponding out-of-sequence customer (denoted by OSC). Let $\underset{\sim}{t}$ be a random variable that represents the interarrival times to the network between the TC and the OSC. Also let $\underset{\sim}{g}|_x$ be the random variable that represents the TC's ISB given his service time x; then by definition

$$h_k(t\mid x)dt = P[\underset{\sim}{g}|_x = k,\ t<\underset{\sim}{t}\leqslant t+dt \mid \underset{\sim}{x}=x]$$

Let 0 be the time of arrival of the tagged customer, and x its given service time. Two cases must be considered according to whether or not the interarrival time is less or greater than x. (See Fig. 4).

i) $t<x$: The probability of the above equation is equal to the probability of two disjoint events:

1) The event of k arrivals in $(0,t)$, and one arrival at $t,t+dt$.

2) The event that the k arrivals depart before x and the arrival at $t,t+dt$ departs after x.

The two events are independent due to the fact that service and arrival processes are independent. The probability of the first event is

$$\frac{(\lambda t)^k}{k!}e^{-\lambda t}\lambda dt$$

As for the second event, we know that the k arrivals generated by a Poisson process in $(0,t)$ are uniformly and independently distributed point processes in $(0,t)$. Hence the probability that they all leave before x is

$$\left[\int_0^t F(x-u)\frac{du}{t}\right]^k$$

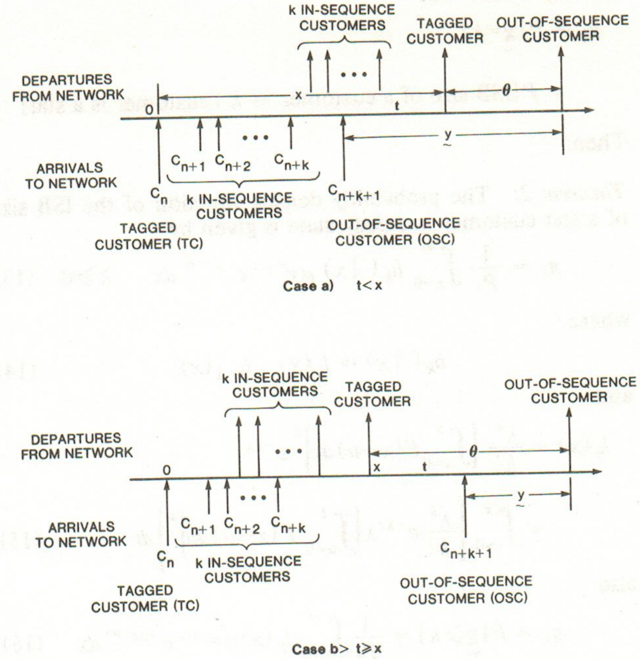

Fig. 4. In-sequence bulks.

Moreover the probability that C_{n+k+1} is an OSC (i.e., that it leaves after x) is

$$(1-F(x-t))$$

The product of the three expressions above is exactly $h_k(t\mid x)$ for $t<x$.

ii) $t\geqslant x$: In this case the two events are:

1) The event of k arrivals in $(0,x)$, zero arrivals in (x,t) and one arrival in $(t,\ t+dt)$

2) The event that the k arrivals leave before x

The probability of the first event is:

$$\frac{(\lambda x)^k}{k!}e^{-\lambda x}e^{-\lambda(t-x)}\lambda dt$$

Similar to the previous case, the probability of the second event is

$$\left[\int_0^x F(x-u)\frac{du}{x}\right]^k$$

The product of the two above expressions yields $h_k(t\mid x)$ for $t\geqslant x$. Finally the marginal distributions of Eqs. (11) and (12) are direct consequences of Eq. (10). Let us note, also, that the proof of Lemma 1 does not make use of the assumption of exponential distribution for the network service time distribution of the selected arbitrary customer since that time is fixed.

In the problem being considered we are mainly interested in finding the ISB size of a star customer, i.e., the selected TC must be a star. Let

$$g_k = P[\underset{\sim}{g}=k]$$

$$= P[\text{ISB size of a customer} = k \mid \text{customer is a star}]$$

Then:

Theorem 2: The probability density function of the ISB size of a star customer in steady state is given by

$$g_k = \frac{1}{P_*}\int_{x=0}^{\infty} h_k(\cdot \mid x)\, \mu e^{-\mu x} e^{-\rho e^{-\mu x}} dx \qquad k \geqslant 0 \qquad (13)$$

where

$$h_k(\cdot \mid x) = l_k(x) - l_{k+1}(x) \qquad (14)$$

and

$$l_k(x) = \frac{\lambda^k}{k!}\left[\int_{u=0}^{x} F(x-u)\,du\right]^k e^{-\lambda x}$$

$$+\int_{t=0}^{x}\left\{\frac{\lambda^k}{k!}e^{-\lambda t}\lambda\left[\int_{u=0}^{t} F(x-u)\,du\right]^k\right\}dt \qquad (15)$$

also

$$q_k = P[\underset{\sim}{g} \geqslant k] = \frac{1}{P_*}\int_{x=0}^{\infty} l_k(x)\mu e^{-\mu x} e^{-\rho e^{-\mu x}} dx \qquad (16)$$

Proof: The proof is obvious because of Lemma 1. The chosen TC is now a star customer and what is needed is the unconditioning of the marginal probability distributions of Eq. (11) with respect to x. Since x is the service time of a star customer its p.d.f. is given by Eq. 5 (Theorem 1).

Because of Lemma 1

$$g_k = \int_0^{\infty} h_k(\cdot \mid x) f_s(x)\,dx \qquad (17)$$

The substitution above of $f_s(x)$ by its expression in Eq. (5) proves Eq. (13). Eq. (15) is a direct consequence of Eqs. (10), (11) and (13) but the proof is omitted here (see Reference [4]). Eq. (16) results directly from the above considerations.

Note that it can be easily checked that $\sum_0^{\infty} g_k = 1$ and that $q_0 = 1$. Moreover, the average ISB size of a star customer is given by:

$$\overline{g} = \sum_{k=1}^{\infty} k g_k = \sum_{k=1}^{\infty} q_k \qquad (18)$$

From Eqs. (16) and (18), and interchanging the integral and summation signs, we find

$$\overline{g} = \frac{1}{P_*}\int_0^{\infty}\left[\sum_{k=1}^{\infty} l_k(x)\right]\mu e^{-\mu x} e^{-\rho e^{-\mu x}} dx \qquad (19)$$

From Eq. (15)

$$1 + \sum_{k=1}^{\infty} l_k(x) = e^{-\lambda x} e^{\lambda \int_0^x F(x-u)\,du} \qquad (20)$$

$$+\int_{t=0}^{x} \lambda e^{-\lambda t} e^{\lambda \int_0^t F(x-u)\,du} dt$$

By the substitution of Eq. (20) into (19) and by integrating, it can be shown that

$$\overline{g} = \frac{1}{P_*}(1-P_*)$$

hence

$$1+\overline{g} = \frac{1}{P_*} \qquad (21)$$

Note that $\overline{G} = 1+\overline{g}$ is the average bulk size, consequently Eq. (21) checks with the results of theorem 1.

□

III.3 Distribution of Interarrival and Interdeparture Times of Star Customers

We now proceed with the derivation of the p.d.f. of the interarrival time to the network and interdeparture time from the network of the star customers in steady state.

Theorem 3: The sequence of interarrival times of star customers to the network and of interdeparture times of star customers from the network converge in steady state to identically distributed random variables, whose p.d.f. is

$$h(t) = \begin{cases} \mu e^{-\mu t} + \dfrac{\rho}{1(-\rho)}\,\dfrac{e^{-\rho}}{(1-e^{-\rho})}[\lambda e^{-\lambda t} - \mu e^{-\mu t}] & \rho \neq 1 \\[2ex] \mu e^{-\mu t}(1+\dfrac{e^{-1}}{1-e^{-1}}(\mu t - 1)) & \rho = 1 \end{cases} \qquad (22)$$

and whose variance is

$$\sigma_{t_s} = \begin{cases} \dfrac{1}{\mu^2}(2-\dfrac{1}{(1-e^{-\rho})^2}) + \dfrac{2}{\lambda^2}\,\dfrac{e^{-\rho}\rho(1+\rho)}{1-e^{-\rho}} & \rho \neq 1 \\[2ex] \dfrac{1}{\mu^2}\left[2(1-\dfrac{e^{-1}}{1-e^{-1}}) - \dfrac{1}{(1-e^{-1})^2}\right] + \dfrac{6}{\mu^3} & \rho = 1 \end{cases} \qquad (23)$$

Proof: First we prove that Eq. (22) holds true for the interarrival times of star customers in the network.

a) Interarrival Times

Similar to the proof of Theorem 2, the p.d.f. of the interarrival times of star customers is a direct consequence of Lemma 1 and Theorem 1. We must however notice that since the TC of Lemma 1 (Fig. 4) is now a star customer then the first out-of-sequence customer thereafter is also a star customer.

From the above considerations and Eqs. (5) and (12)

$$h(t) = \int_0^{\infty} h(t \mid x) f_s(x)\,dx \qquad (24)$$

Eq. (24) and the results of Lemma 1 and Theorem 1 yield Eqs. (22) and (23). The algebra is omitted here.

We can easily check that $\int_0^{\infty} h(t)\,dt = 1$. Also let $\overline{t_s}$ be the average time between arrival of star customers; then

$$\overline{t_s} = \int_0^{\infty} t h(t)\,dt$$

After some algebra, we find

$$\overline{t_s} = \frac{1}{\lambda P_*}$$

Recall that $\lambda P_* = \lambda_s$ is the rate of arrival of star customers, hence the equation above gives the expected result.

b) Interdeparture times

Let $\underset{\sim}{\theta}$ be the interdeparture time between the two star customers of Fig. 4 (TC and OSC are star customers), and $l(\theta)$ the p.d.f. of $\underset{\sim}{\theta}$. If we condition on the service time of the TC star customer then

$$l(\theta) = \int_0^\infty l(\theta \mid x) f_s\, dx \qquad (25)$$

where

$$l(\theta \mid x) = P[\theta < \underset{\sim}{\theta} \leqslant \theta + d\theta \mid \text{service time of TC star} = x] \qquad (26)$$

In order to determine $l(\theta \mid x)$ we further need to condition on the interarrival times t between the two star customers, TC and OSC. Let

$$L(\theta \mid x,t) = P[\underset{\sim}{\theta} \leqslant \theta \mid x_x = x,\ \underset{\sim}{t} = t] \qquad (27)$$

Let $\underset{\sim}{y}$ denote the service time of the OSC, then

$$\underset{\sim}{\theta} = \underset{\sim}{y} + \underset{\sim}{t} - x_s \qquad (28)$$

thus

$$L(\theta \mid x,t) = P[\underset{\sim}{y} \leqslant \theta + x - t] \qquad (29)$$

a. $t > x$: then $\underset{\sim}{y}$ is just the service of an arbitrary customer

b. $t < x$: then $\underset{\sim}{y}$ is further known as being greater than $x - t$

Consequently

$$L(\theta \mid x,t) = \begin{cases} P[\underset{\sim}{y} \leqslant \theta + x - t \mid \underset{\sim}{y} > x - t] & t < x \\ P[\underset{\sim}{y} \leqslant \theta + x - t] & t \geqslant x \end{cases} \qquad (30)$$

Where $\underset{\sim}{y}$ is the service time of an arbitrary customer (i.e., it is exponentially distributed). Furthermore $\theta + x - t$ must be positive for $L(\theta \mid x,t)$ to exist; thus for $\theta \geqslant 0$

$$L(\theta \mid x,t) = \begin{cases} 1 - e^{-\mu\theta} & t < x \\ 1 - e^{-\mu(\theta + x - t)} & x \leqslant t \leqslant \theta + x \\ 0 & t > \theta + x \end{cases} \qquad (31)$$

Let $l(\theta \mid x,t)$ be the p.d.f. that corresponds to the probability distribution function, $L(\theta \mid x,t)$, i.e.,

$$l(\theta \mid x,t) = \frac{\partial}{\partial\theta} L(\theta \mid x,t)$$

Thus

$$l(\theta \mid x,t) = \begin{cases} \mu e^{-\mu\theta} & t < x \\ \mu e^{-\mu(\theta + x - t)} & x \leqslant t \leqslant \theta + x \\ 0 & t > \theta + x \end{cases} \qquad (32)$$

is the p.d.f. of the time between departures knowing $\underset{\sim}{x}$ and t. In [4] we find the p.d.f of $\underset{\sim}{t}$ knowing $\underset{\sim}{x}$, thus

$$l(\theta \mid x) = \int_{t=0}^{\theta + x} l(\theta \mid x,t)\, h(t \mid x)\, dt \qquad (33)$$

The next step (omitted here) is to show that [4]

$$l(\theta \mid x) = \begin{cases} \mu e^{-\mu\theta}[1 - e^{-\rho} e^{\rho e^{-\mu x}} + \dfrac{\rho e^{-\rho}}{1-\rho} e^{\rho e^{-\mu x}} (e^{(\mu-\lambda)\theta} - 1)] & \rho \neq 1 \\ \mu e^{-\mu\theta}(1 - e^{-\rho} e^{\rho e^{-\mu x}} + \lambda e^{-\rho} e^{\rho e^{-\mu x}} \theta) & \rho = 1 \end{cases} \qquad (34)$$

Using Eqs. (25) and (34) it can then be shown that $l(t) = h(t)$ (Eq. (22)).

□

So far we have proven that, given an arbitrary tagged customer which is a star customer, the interarrival time and interdeparture time of the next arbitrary customer who is a star customer are identically distributed and their common p.d.f. is given by Eq. (22). In order for the *sequence* of interarrival times and interdeparture times of star customers to be identically distributed it is enough to prove that the network service time distribution of that next star (OSC) is identical to that of the original star customer. This comes about from the following considerations: From Lemma 1 we know that $h_k(t \mid x)$ depends only on the given service time of the tagged customer. If that customer is a star customer then the unconditioning must be done according to Eq. (17). Consequently if we prove that the p.d.f. of the service time of the next star in sequence is identical to $f_s(x)$ then the application of Eq. (17) yields the same distribution $h(t)$. A similar demonstration is obviously true for the p.d.f. of the interdeparture time between the next star and the one following him. The proof that the pd.f. of the network service time of the next star is identical to that of the original star is omitted here (see [4]).

Lastly we note that the interdeparture time $\underset{\sim}{\theta}$ and $\underset{\sim}{x}_s$ are dependent random variables because of Eq. (32) and also that $\underset{\sim}{g}$ the ISB size and $\underset{\sim}{x}_s$ are dependent random variables because of Eq. (13). Consequently $\underset{\sim}{\theta}$ and $\underset{\sim}{g}$ are dependent random variables. This is a major problem in the analysis of the database station queueing system since it cannot be modelled as an M/G/1 with bulk arrivals.

III.4 Distribution of the Waiting Time of a Non-Star Customer

Theorem 4:

The probability distribution function of the waiting time in the virtual box of a *non-star* customer is given by:

$$F(w) = P[\underset{\sim}{w} \leqslant w \mid \text{customer is not a star}]$$

$$= \frac{(1 - e^{-\rho e^{-\mu w}}) e^{\mu w} - 1 + e^{-\rho}}{\rho - 1 + e^{-\rho}} \qquad w \geqslant 0 \qquad (35)$$

Proof: For the sake of brevity the proof is omitted here but can be found in [4].

It is interesting to check that $F(0) = 0$ and $F(\infty) = 1$. The first equality is obvious from Eq. (35). Whereas to prove the second equality we notice that: $e^{-\mu w}$ becomes very small when w becomes large. Consequently let us expand the numerator of Eq. (35),

$$e^{\mu w}[1-1+\rho e^{-\mu w}-\frac{\rho^2}{2}e^{-2\mu w}+o(e^{-\mu w})]-1+e^{-\rho}=$$

$$\rho-\frac{\rho^2}{2}e^{-\mu w}-1+e^{-\rho}+o(e^{-\mu w})$$

If we take the limit when $w \to \infty$ we obtain $\rho - 1 + e^{-\rho}$ which is exactly the same as the denominator.

III.5 Distribution of the Number of Star Customers in the Network

Theorem 5: Let $P_{n,k} = P[k$ star customers in service $| n$ customers in service$]$ and and

$$G_n(z) = \sum_k P_{n,k} z^k$$

Then

$$G_n(z) = \frac{1}{z+n}\binom{z+n}{n} \tag{36}$$

and the average number of star customers given n is

$$\bar{k}_n = 1+\frac{1}{2}+\cdots+\frac{1}{n} \tag{37}$$

Proof: Obviously $P_{n,0} = 0$ and $P_{n,k} = 0$ if $k > n$. Also if $n=1$, then $P_{1,1}=1$. Let us consider the case where $n \geq 2$, let C_{n_n} be the customer with the *highest* number among the n in service and let $(n_1, n_2, \cdots, n_{n-1})$ be the numbers of the $(n-1)$ left in increasing order. For C_{n_k} to be a star customer he must be such that his service time is greater than that of $C_{n_1}, \cdots, C_{n_{k-1}}$. Thus the number of stars among $(n_1, \cdots, n_{n-1})$ is independent of C_{n_n}. Two cases are possible in order to obtain k star customers

1) C_{n_n} is a star with probability $\frac{1}{n}$ and there are $k-1$ star customers among $(n_1, \cdots, n_{n-1})$.

2) C_n is not a star and there are k star customers among $(n_1, \cdots, n_{n-1})$.

Thus

$$P_{n,k} = \frac{1}{n}P_{n-1,k-1} + \frac{n-1}{n}P_{n-1,k} \qquad n \geq 2 \quad k < n \tag{38}$$

If we multiply by z^k and sum over k from 1 to infinity we find

$$G_n(z) = \frac{z+n-1}{n}\, G_{n-1}(z) \qquad n \geq 2 \tag{39}$$

note that $G_1(z) = z$. The last two equations above yield Eq. (36).

Let $\overline{k_n}$ be the average number of customers, then

$$\overline{k_n} = \sum_{k=1}^{\infty} k\, P_{n,k} = G'_n(1)$$

Taking the derivative with respect to z in Eq. (36), we arrive at

$$G'_n(1) = \frac{1}{n} + G'_{n-1}(1) \qquad n \geq 2$$

also

$$G'_1(1) = 1$$

The above equations yield Eq. (37) which is a harmonic series. This result can also be obtained directly from Eq. (38).

Similar results to Theorem 5 have also been found in the study of time complexity of SITU permutation algorithms by Knuth [5]* where the equivalent of Knuth's LOO (left out of order) number is the star customer here.

III.6 Limiting Behavior

Two important cases will be studied: $\rho \to 0$ and $\rho \to +\infty$.

Corollary 1:

At the limit, when $\rho \to 0$ the following properties are true:

$$\begin{aligned} &P_* \to 1 && (a)\\ &f_s(x) \to \mu e^{-\mu x} && (b)\\ &\left.\begin{aligned} &g_k \to 0 \quad k \geq 1\\ &g_0 \to 1 \end{aligned}\right\} && (c)\\ &h(t) \to \lambda e^{-\lambda t} && (d)\\ &F(w) \to 1 - e^{-\mu w} && (e) \end{aligned} \tag{40}$$

Proof:

Eqs. (40a), (40b), and (40d) are obvious. Eq. (40c) is also obvious since

$$\sum_{k=1}^{\infty} k g_k = \bar{g} = \frac{1}{P_*} - 1$$

Thus

$$\lim_{\rho \to 0} \bar{g} = 0 \quad \text{which yields} \quad \lim_{\rho \to 1} g_k = 0 \quad k \geq 1$$

Eq. (40c) can also be proved directly from Eq. (16) (see [4]). In order to prove Eq. (40e) let us expand the numerator and denominator in Eq. (35) around ρ

$$F(w) = \frac{(1-1+\rho e^{-\mu w} - \frac{\rho^2}{2}e^{-2\mu w})e^{\mu w} - 1 + 1 - \rho + \frac{\rho^2}{2} + o(\rho^2)}{\rho - 1 + 1 - \rho + \frac{\rho^2}{2} + o(\rho^2)}$$

$$F(w) = \frac{-\frac{\rho^2}{2}e^{-\mu w} + \frac{\rho^2}{2} + o(\rho^2)}{\frac{\rho^2}{2} + o(\rho^2)}$$

Taking the limit when $\rho \to 0$ gives Eq. (40e).

* We are grateful to Dr. Mickey Krieger who pointed out this work by Knuth.

Let us note that the results of Eq. (40) show that at the limit when $\rho \to 0$ very little disorder occurs in the system. In fact at the limit all customers are star customers. Moreover, if an inversion of order occurs at the limit, it will be between exactly two customers. Thus the higher number will depart first and the second will spend his remaining time in the network and then will depart as a star customer. That remaining time is the eligibility waiting time for the non-star customer, and it is exponentially distributed. This explains Eq. (40e).

Corollary 2

At the limit when $\rho \to \infty$ the following properties are true:

$$P_* \to 0 \quad (a)$$
$$\overline{G} \to \infty \quad (b) \qquad (41)$$
$$h(t) \to \mu e^{-\mu t} \quad (c)$$

The proof of the above is obvious. Furthermore let us note that

$$\lim_{\rho \to \infty} f_s(x) = 0 \qquad x \text{ belongs to } [0,\infty) \; (x \text{ finite})$$

which indicates that the p.d.f. becomes an impulse at infinity.

Also

$$\lim_{\rho \to \infty} F(w) = 0 \qquad w \in [0,\infty)$$

thewhich also indicates that the p.d.f. of the waiting time of non-star customers is an impulse at infinity. Therefore the waiting time at the virtual waiting box becomes infinite. This checks with the previous result whereby the network service time of a star customer becomes infinite when $\rho \to \infty$. Furthermore we note that the departure process of star customers is a *Poisson Process* with a rate equal to the network service rate μ. This is a quite surprising result. Star customers spend on the average an infinite time in the network but depart at a rate μ.

IV. Numerical Examples

In this section we show some numerical examples illustrating the analytic results developed in the preceding sections.

Figure 5 shows the variation of P_*, the probability that an arbitrary customer is a star customer, versus the network traffic intensity $\rho = \lambda/\mu$. We notice that P_* drops relatively rapidly in the region centered at $\rho=1$. This is intuitive since a customer is *not* a star if it arrives while at least one other customer is in the network and departs earlier than one of these customers. Since ρ is the mean number of customers seen in the network at an arrival instant, for $\rho<1$ there is small probability of any customers in the network at an arrival instant (and an even smaller probability that the arriving customer will depart earlier than a customer found in the network). For $\rho >> 1$, P_* is approximately $1/(\rho+1)$ which is intuitive because an arriving customer sees an average of ρ customers in the network and each is equally likely to be the last to depart.

Figure 6 shows the probability density function of the network service time of a star customer as a function of ρ with $\mu=2$. Figure 7 shows the eligibility waiting time of a non-star customer as a function of ρ. For $\rho=0$ these density

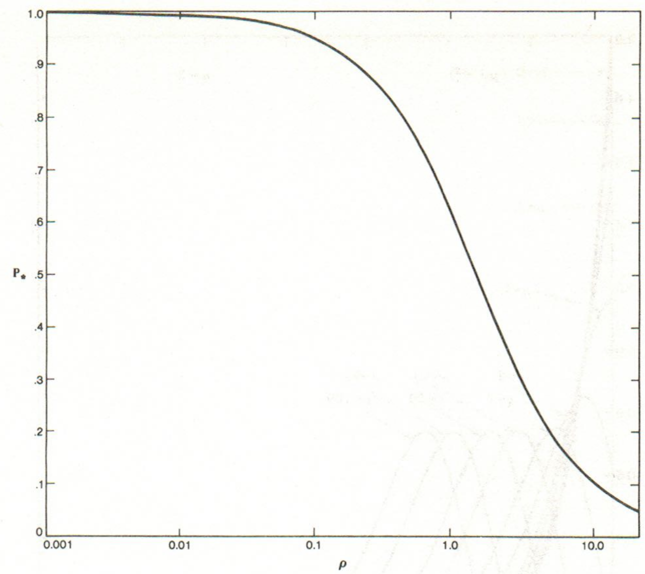

Fig. 5. Probability of a star customer.

functions are exponential. For high ρ they are unimodal density functions which are of approximately constant shape but the mode (and mean) is shifting toward infinity.

It was noted in the sections describing the limiting behavior of the infinite server that at both extremes ($\rho \to 0$ and $\rho \to \infty$) the departure process of star customers from the network is Poisson. In Figure 8 the coefficient of variation of the inter-departure times of star customers is plotted versus ρ. At both low and high values of ρ the coefficient of variation is close to 1 and around $\rho=1$ there is a noticeable dip. Interest in this departure process comes from a desire to approximate the database manager queue as an M/G/1 queue with bulk arrivals. While the evidence so far indicates that the Poisson arrival assumption is reasonable, there is still the problem that bulk size interarrival times are correlated. Dealing with this problem is a topic for further research.

Figure 9 shows the variation of the mean network service time of a star customer and the mean eligibility waiting time of a non-star customer as a function of ρ. We note that these means are relatively constant until the region $\rho=1$, at which point they increase markedly. No simple relationship between these two values has been found although it is suggested by the data the such a relationship may well exist.

Conclusion

In this paper we have presented some basic results relating to the disorder introduced when ordered arrivals are subjected to delays in an M/M/∞ server. The motivation for examining this behavior is ultimately to be able to develop models for distributed database systems which would include the effect of concurrency control algorithms. This goal is not met in this paper but rather we have presented some initial results which are of some interest in themselves and which will be useful in further development of a total system model.

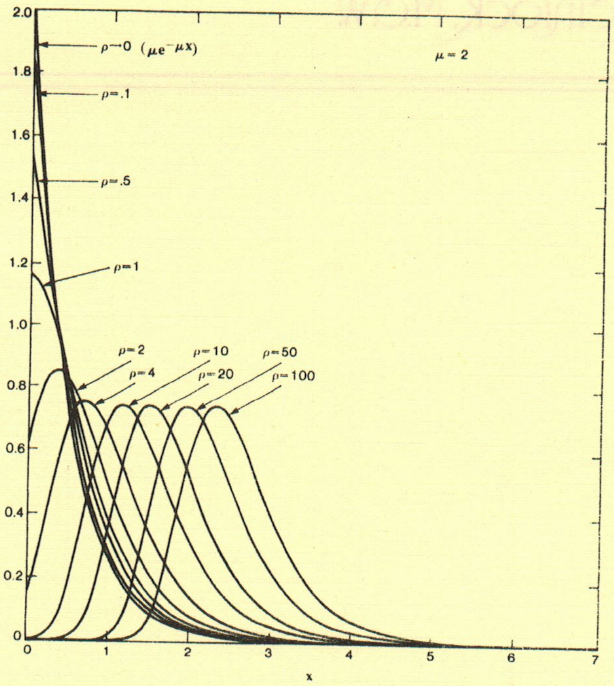

Fig. 6. Probability density function of service time of a star customer.

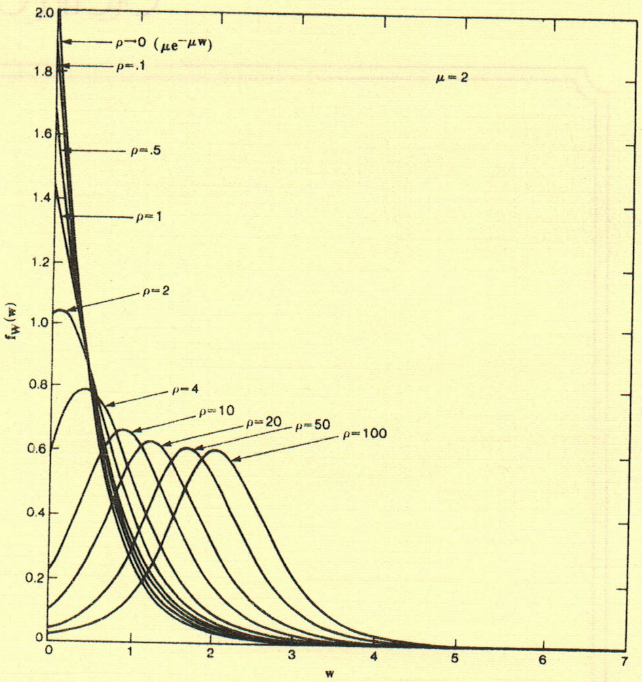

Fig. 7. Probability density function of the eligibility waiting time of a non-star customer.

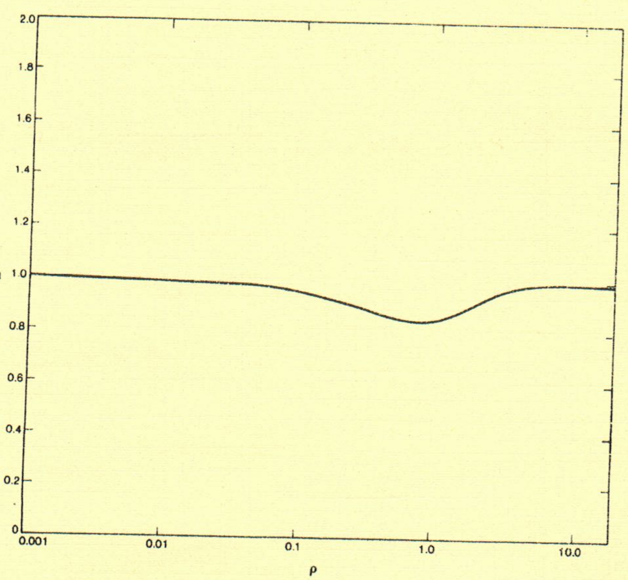

Fig. 8. Coefficient of variation of the interdeparture times of star customers from the network.

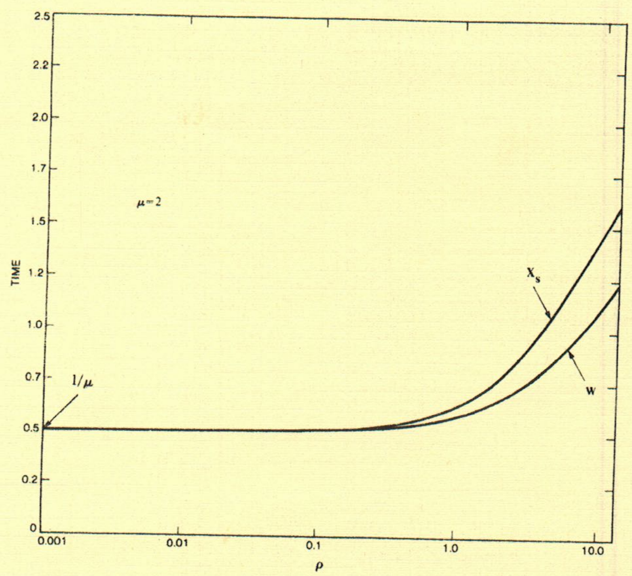

Fig. 9. Average service time of a star customer, $\overline{X}_s$ Average eligibility waiting time, W.