OPTIMUM BRIBING FOR QUEUE POSITION*

Leonard Kleinrock

University of California, Los Angeles, California

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In this paper we consider that relative position in queue is determined according to the size of a customer's bribe (which is paid before the customer sees the queue length). Such a policy allows the customer himself to affect his own queue position, rather than the classical approach of assuming that a customer is preassigned to some (possibly continuous) priority class. For the case of Poisson arrivals, arbitrary service time distribution, and arbitrary distribution of customer bribe, we obtain the average waiting time for customers as a function of their bribe. We consider both preemptive and nonpreemptive disciplines. Examples are presented for various bribing distributions, which demonstrate that many well-known priority queuing systems are special cases of this bribing situation. Furthermore, a cost function is defined after we introduce the notion of an impatience factor (which converts seconds of wait into dollars). Conditions for optimum bribing are then determined, where the optimization refers to minimizing the average cost subject to a mean bribe constraint. An example for exponential service and exponential bribing is carried out and the results are plotted.

A NUMBER of priority queuing disciplines have been studied in the past (for example: head of the line, COBHAM;[1] random ordering, VAULOT;[2] last-come first-served, WISHART;[3] delay dependent, KLEINROCK;[4] head-of-the-line preemptive, WHITE AND CHRISTIE[5]). In these earlier studies, the relative priority given to any customer was completely out of his individual control; the customer, in effect, had no choice as to which priority group he must join.

In this study, we shift the emphasis somewhat, and allow each entering unit to 'buy' his relative priority by means of a bribe. The size of the bribe will be determined, in general, from certain economic factors inherent in the population of customers; in particular, the greater the wealth of a customer, and the greater his dislike of waiting on queue, the greater will be his bribe.

THE MODELS

WE CONSIDER two models, the first of which is a nonpreemptive queuing model. We assume that we have Poisson arrivals at a mean rate of λ

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customers per second. The single-channel service facility has an arbitrary cumulative service time distribution $F(\tau)$ with a mean service time $1/\mu$ sec. Let the customer's bribe, x, be a random variable with an arbitrary cumulative distribution function $B(x)$. We assume that the arrival time, the service time, and the bribe are all independent random variables for each customer and are independent of the values chosen for all other customers.

The system operates as follows: A new arrival to the system offers a nonnegative bribe* x to the 'queue organizer.' This customer is then placed in position on the queue so that all those customers whose bribes $x' \geqq x$ are in front of him and all these with $x'' < x$ are behind him. Newly entering customers may therefore be placed in front of, or behind this customer, depending upon their bribe. Each time the service facility completes work on some customer (who then leaves the system), it then accepts into service the customer at the front of the queue. Once in service, a customer cannot be ejected until he is completely serviced.

In the preemptive case (our second model), we restrict ourselves to a service time distribution $F(\tau) = 1 - e^{-\mu\tau}$ with a preemptive resume rule. In this mode, a customer will be ejected from service if a newly entering unit offers a bribe larger than the bribe he offered.

In both cases, we have

λ = average arrival rate of customers (Poisson distribution).

$1/\mu$ = average service time (arbitrary distribution in the nonpreemptive model, exponential distribution otherwise).

$B(x)$ = cumulative distribution (arbitrary) for bribes.

Whenever customers give identical bribes, they are serviced in a first-come first-served order.

We define, for $\epsilon > 0$, the left and right limits of $B(x)$ as

$$B(x-) = \lim_{\epsilon\to 0} B(x-\epsilon),$$

$$B(x+) = \lim_{\epsilon\to 0} B(x+\epsilon).$$

AVERAGE WAITING TIMES

Case 1. Nonpreemptive rule:

Let $W(x) \equiv$ average waiting time (in queue) for a customer whose bribe is x.

THEOREM 1.

$$W(x) = W_0/[1-\rho+\rho B(x+)][1-\rho+\rho B(x-)], \tag{1}$$

* This bribe may either be thought of as given before the customer sees the length of the queue [in which case the distribution $B(x)$ reflects his measure of wealth and impatience] or as being chosen from $B(x)$ independent of queue length.

where $\rho=\lambda/\mu=$ *utilization factor*

$W_0\equiv$ *expected time to finish the customer found in service upon entry of a new customer to the system*

and

$$W_0=\frac{\lambda}{2}\int_0^\infty \tau^2\,dF(\tau). \tag{2}$$

At those x for which $B(x)$ is continuous we see that equation (1) reduces to

$$W(x)=W_0/[1-\rho+\rho B(x)]^2. \tag{3}$$

Proof. Let us consider a customer (say, the tagged customer) who gives a bribe of size x. The average waiting time in queue, $W(x)$ for this customer, may be calculated as follows. The tagged customer must, on the average, wait a time W_0 before the customer who is in service upon his arrival is finished.* In addition, he must wait until service is given to all those customers still in queue who arrived before he did and whose bribes equaled or exceeded his. The expected number of such customers whose bribes lie in the region $(y, y+dy)$ is

$$\lambda(y)\,W(y)\,dy, \tag{4}$$

where

$$\lambda(y)=\lambda[dB(y)/dy]. \tag{5}$$

Each such unit causes the tagged unit to wait an average time of $1/\mu$ sec. Equation (4) follows from the observation (*see* LITTLE[2]) that the expected number of units in a system is equal to the product of their arrival rate and the expected time they spend in the system. Furthermore, the tagged unit must wait until service is given to all those customers who enter the system while the tagged unit is on the queue and whose bribes exceed his. The expected number whose bribes lie in the interval $(y, y+dy)$ and which arrive during his average wait $W(x)$ is

$$\lambda(y)\,W(x)\,dy. \tag{6}$$

Each such unit adds $1/\mu$ sec to the tagged unit's average wait. Combining these three contributions to the tagged unit's average wait, we get†

$$W(x)=W_0+\int_{x-}^{\infty}\frac{\lambda(y)W(y)\,dy}{\mu}+\int_{x+}^{\infty}\frac{\lambda(y)\,W(x)\,dy}{\mu}, \tag{7}$$

or

$$W(x)=\left[W_0+\int_{x-}^{\infty}\frac{\lambda}{\mu}W(y)\,dB(y)\right]\bigg/\left[1-\int_{x+}^{\infty}\frac{\lambda}{\mu}\,dB(y)\right]. \tag{8}$$

* See for example, SAATY,[6] Sec. 11–21a.

† The lower limits of $x-$ and $x+$ come about since all ties are broken on a first-come first-served basis.

Since $B(\infty)=1$, we have

$$W(x)=\left[W_0+\rho\int_{x-}^{\infty}W(y)\,dB(y)\right]\bigg/[1-\rho+\rho B(x+)]. \tag{9}$$

Replacing $W(x)$ in this last equation by the expression given in equation (1), we find that the theorem will be established if we can prove the following equality:

$$\frac{1}{1-\rho+\rho B(x-)}-1=\rho\int_{x-}^{\infty}\frac{dB(y)}{[1-\rho+\rho B(y+)][1-\rho+\rho B(y-)]}. \tag{10}$$

Let us define

$$A(y)=1-\rho+\rho\, B(y). \tag{11}$$

Let $x_k(k=1, 2, 3, \cdots, K)$ be those values of x at which $B(x)$ has its discontinuities (at most a countable number), and let $\Delta B(x_k)$ be the magnitude of these discontinuities. Further, let $B_1(x)$ be the continuous portion of $B(x)$. We may then express the integral $I(x)$ in equation (10) as

$$\begin{aligned}I(x)=\int_{x-}^{\infty}\frac{dB(y)}{A(y+)A(y-)}=\sum_{x_k=x_{k_0}}^{x_K}\frac{\Delta B(x_k)}{A(x_k-)A(x_k+)}\\+\sum_{x_k=x_{k_0}}^{x_K}\int_{x_k}^{x_{k+1}}\frac{dB_1(y)}{A^2(y)}+\int_{x}^{x_{k_0}}\frac{dB_1(y)}{A^2(y)},\end{aligned} \tag{12}$$

where $x_{k_0}=\min_{x_k\geqq x}x_k$ and $x_{K+1}\equiv\infty$. For any open interval (α, β) that contains no points of discontinuity of $B(y)$, we see that $B_1(y)=B(y)$; thus, letting $z=B_1(y)$, we get

$$\int_{\alpha}^{\beta}\frac{dB_1(y)}{A^2(y)}=\int_{B_1(\alpha)}^{B_1(\beta)}\frac{dz}{(1-\rho+\rho z)^2}=\frac{B_1(\beta)-B_1(\alpha)}{A(\beta)A(\alpha)}. \tag{13}$$

For such an interval we observe that

$$[B_1(\beta)-B_1(\alpha)]/A(\beta)A(\alpha)=[B(\beta-)-B(\alpha+)]/A(\beta-)A(\alpha+); \tag{14}$$

thus, we get

$$\begin{aligned}I(x)=\textstyle\sum_{x_k=x_{k_0}}^{x_K}\ &\{[\Delta B(x_k)/A(x_k-)A(x_k+)]\\&+[B(x_{k+1}-)-B(x_k+)]/A(x_{k+1}-)A(x_k+)\}\\&+[B(x_{k_0}-)-B(x-)]/A(x_{k_0}-)A(x-).\end{aligned} \tag{15}$$

Since $\Delta B(x_k)=B(x_k+)-B(x_k-)$, and since $B(x)=(1/\rho)[A(x)-1+\rho]$ from equation (11), we have

$$I(x)=(1/\rho)\textstyle\sum_{x_k=x_{k_0}}^{x_K}\{[1/A(x_k-)]-[1/A(x_{k+1}-)]\}$$

$$+(1/\rho)\{[1/A(x-)]-[1/A(x_{k_0}-)]\} \quad (16)$$

$$=(1/\rho)\{[1/A(x-)]-[1/A(x_{K+1}-)]\}.$$

Now, since $B(\infty)=1$,

$$A(x_{K+1}^{-})=1-\rho+\rho B(\infty)=1.$$

Thus, equation (16) becomes

$$I(x)=(1/\rho)\{[1/A(x-)]-1\}. \quad (17)$$

Using equation (17) in equation (10), we get

$$\{[1/[1-\rho+\rho B(x-)]\}-1=\rho I(x)=[1/A(x-)]-1.$$

Thus, this last identity establishes that equation (1) is indeed the solution for $W(x)$ as defined in equation (9).

Case 2. Preemptive rule:

In this case we require that $F(\tau)=1-e^{-\mu\tau}$. Let

$T(x)$ = average time spent in the system (queue plus service) for a customer whose bribe is x.

THEOREM 2.

$$T(x)=(1/\mu)/[1-\rho+\rho B(x+)][1-\rho+\rho B(x-)]. \quad (18)$$

At those x for which $B(x)$ is continuous, we see that equation (18) reduces to

$$T(x)=(1/\mu/)[1-\rho+\rho B(x)]^2. \quad (19)$$

Proof. The proof here is almost identical to that of Theorem 1. Instead of equation (7), we get

$$T(x)=\frac{1}{\mu}+\int_{x-}^{\infty}\frac{\lambda(y)T(y)}{\mu}\,dy+\int_{x+}^{\infty}\frac{\lambda(y)T(x)}{\mu}\,dy. \quad (20)$$

In this case, the expected additional time to service units that have been preempted is still $1/\mu$ due to the memoryless property of the exponential distribution $F(\tau)$. From equation (20), we get

$$T(x)=\left[\frac{1}{\mu}+\rho\int_{x-}^{\infty}T(y)\,dB(y)\right]\Big/\,[1-\rho+\rho B(x+)]. \quad (21)$$

We now try the solution given in equation (18), which results in the following equation

$$\frac{1}{1-\rho+\rho B(x+)}-1=\rho\int_{x-}^{\infty}\frac{dB(y)}{[1-\rho+\rho B(y+)][1-\rho+\rho B(y-)]}.$$

But this is the same as equation (10) which we have already proven.

Thus, equation (18) is indeed the solution to equation (21), which completes the proof of Theorem 2.

We comment that in both cases studied above, we may make use of the expression given below, which relates the average wait in queue and the average wait in queue plus service:

$$T(x)=W(x)+1/\mu.$$

DISCUSSION AND EXAMPLES OF AVERAGE WAITING TIMES

We begin by observing that in the special case

$$B(x)=\begin{cases}1 & x\geqq x_0,\\ 0 & x<x_0,\end{cases} \tag{22}$$

we have

Case 1: $$W(x_0)=W_0/(1-\rho), \tag{23}$$

Case 2: $$T(x_0)=(1/\mu)/(1-\rho). \tag{24}$$

This $B(x)$ corresponds to the classical first-come first-served queue since all bribes are the same resulting in no effective bribe at all. Thus equations (23) and (24) should correspond to the well-known results for $M/G/1$ which they do.

In the case where $B(x)$ is continuous at the origin, giving $B(0)=0$, we see that

Case 1: $$W(0)=W_0/(1-\rho)^2, \tag{25}$$

Case 2: $$T(0)=(1/\mu)/(1-\rho)^2. \tag{26}$$

The behavior at zero bribe should describe the waiting time for the lower priority group of a two-priority class head-of-the-line system where the arrival rate of this low priority group is negligible compared to the total arrival rate. Indeed, as can be seen from references 1 and 5, the equations above are consistent.

When only a finite (countable) set of bribes are allowed (at the values x_k), then we have a discrete distribution that yields

Case 1:

$$W(x_k)=W_0/[1-\rho\textstyle\sum_{i=k+1}^{i=K}\ \Delta B(x_i)][1-\rho\sum_{i=k}^{i=K}\ \Delta B(x_i)], \tag{27}$$

Case 2:

$$T(x_k)=(1/\mu)/[1-\rho\textstyle\sum_{i=k+1}^{i=K}\ \Delta B(x_i)][1-\rho\sum_{i=k}^{i=K}\ \Delta B(x_i)]. \tag{28}$$

These last two equations correspond respectively to the results for Cobham's[1] head-of-the-line system, and the preemptive head-of-the-line system studied by White and Christie.[5]

When $B(x)$ is a purely continuous distribution, then $T(x)$ (Case 2) is similar to a model of continuous priorities studied by PHIPPS.[8]

We also note that the average waiting times given in Theorems 1 and 2 obey the Conservation Law.[9] We show this for equation (1) only since equation (18) is of the same form. The Conservation Law (in its continuous form) states that, for $0 \leqq \rho < 1$

$$\int_0^\infty \rho(x)\, W(x)\, dx = \frac{\rho}{1-\rho} W_0.$$

Using equations (1), (11), (12), and (17) we have

$$\begin{aligned}\int_0^\infty \rho(x)\, W(x)\, dx &= \int_0^\infty \rho \frac{W_0\, dB(x)}{[1-\rho+\rho B(x+)][1-\rho+\rho B(x-)]} \\ &= \rho W_0 \int_0^\infty \frac{dB(x)}{A(x+)A(x-)} \\ &= \rho W_0 I(0) \\ &= \rho W_0 \frac{1}{\rho}\left[\frac{1}{A(0-)} - 1\right].\end{aligned}$$

But $A(0-) = 1-\rho+\rho B(0-) = 1-\rho$.

Thus

$$\begin{aligned}\int_0^\infty \rho(x)\, W(x)\, dx &= W_0\left[\frac{1}{1-\rho} - 1\right] \\ &= W_0 \frac{\rho}{1-\rho},\end{aligned}$$

which establishes the required conservation relation.

OPTIMUM BRIBING

AS SOON AS we introduce the notion of a bribe, we must then consider other cost factors as well. In particular, we define an impatience factor $\alpha(\geqq 0)$ that measures how many dollars* it costs a customer for each second that he spends in the system. We may use one of two definitions of the system, i.e., the system may be defined as the queue alone or it may be defined as the queue plus the service facility.

Considering first, the case of waiting time in queue, we define the cost function $C(\alpha)$ as

$$C(\alpha) = x_\alpha + \alpha W(x_\alpha), \tag{29}$$

where, again,

* This cost may be measured in terms of customer inconvenience or impatience, if you will.

α = customer's impatience factor,

x_α = bribe offered by this customer (given α), and

$W(x_\alpha)$ = average waiting time (in queue) for a customer whose bribe is x_α.

Thus, $C(\alpha)$ is the sum of the customer's bribe (in dollars) and his cost of waiting (in dollars). We assume that customers have (self-) assigned values of α before they enter the system, and that the population of customers, as a whole, produce a probability distribution $P(\alpha)$ on the random variable α, i.e.,

$P(\alpha)$ = Probability that an entering customer has an impatience factor $\alpha' \leqq \alpha$.

The queuing models here are the same as those considered earlier where now, the bribe x_α is some (deterministic) function of the random variable α. We have thus shifted emphasis from the situation in which a customer offers a random bribe to a situation where the customer's bribe is functionally related to his (random) impatience factor α.

We pose the following optimization problem: Find that function, x_α, which minimizes the expected cost C, i.e.,

$$\underset{x_\alpha}{\text{minimize}}\left[C \equiv \int_0^\infty C(\alpha)\, dP(\alpha)\right], \tag{30}$$

subject to an average bribe constraint equal to B, i.e.,

$$B = \int_0^\infty x_\alpha\, dP(\alpha). \tag{31}$$

The solution to this problem is contained in the following theorem.

THEOREM 3. *For any $P(\alpha)$, the function x_α will be an optimum bribing function if and only if x_α is a strictly increasing function of α (for all α outside a set S having the property $\int_S dP(\alpha) = 0$).*

Proof. We must choose x_α to minimize

$$C = \int_0^\infty [x_\alpha + \alpha W(x_\alpha)]\, dP(\alpha). \tag{32}$$

Due to equation (31), this is equivalent to minimizing

$$C - B = \int_0^\infty \alpha W(x_\alpha)\, dP(\alpha).$$

Define

$$\rho(\alpha) = \rho[dP(\alpha)/d\alpha]. \tag{33}$$

We may interpret the quantity $\rho(\alpha)\, d\alpha$ as the fraction of time that the server is busy serving customers whose impatience factor lies in the interval $(\alpha, \alpha+d\alpha)$. Using equation (33) we then find that for $0<\rho$,

$$C-B=\frac{1}{\rho}\int_0^\infty \alpha\rho(\alpha)\, W(x_\alpha)\, d\alpha. \tag{34}$$

The Conservation Law[9] may be rewritten as follows

$$\int_0^\infty \rho(\alpha)W(x_\alpha)\, d\alpha=\frac{\rho}{1-\rho}\, W_0. \tag{35}$$

We note that minimizing equation (34) involves finding that function x_α such that the product of the function $\rho(\alpha)W(x_\alpha)$ and α has a minimum area. However, equation (35) states that the first of these functions must itself have a *constant* area. Since $\rho(\alpha)$ is independent of x_α, we must look for conditions on $W(x_\alpha)$. Clearly, since $W(x_\alpha)\geqq W_0$ (since $A(x+)Ax-)>0$), a necessary* condition on $W(x_\alpha)$ is that

$$dW(x_\alpha)/d\alpha<0 \tag{36}$$

for all $\alpha \notin S$. This last condition comes about since we must weight $\rho(\alpha)W(x_\alpha)$ by the strictly increasing function α. Condition (36) may be rewritten as

$$[dW(x_\alpha)/dx]/(d\alpha/dx)<0. \tag{37}$$

Since $B(x)$ is a nondecreasing function, $dB(x)/dx\geqq 0$. Thus, from equation (1) and equation (18) and since $A(x+)A(x-)>0$, we get that

$$\begin{aligned} dW(x_\alpha)/dx_\alpha&<0 \text{ at those } x \text{ for which } dB(x)/dx>0,\\ &=0 \text{ at those } x \text{ for which } dB(x)/dx=0. \end{aligned} \tag{38}$$

From equations (37) and (38) then, we have that our necessary condition on x_α becomes

$$dx_\alpha/d\alpha>0 \quad \text{for} \quad \alpha \notin S. \tag{39}$$

That this last is achievable is obvious for a large family of functions (e.g., $x_\alpha=\alpha$). From this family, however, we may use only those functions satisfying equation (31), but such functions clearly exist (e.g., *see* the example in the next section).

Consider an interval $\alpha_1<\alpha<\alpha_2$ in which $P(\alpha)$ is constant. Clearly x_α can be arbitrary in any such interval without affecting C; the same is true at any point α for which $P(\alpha)$ is continuous. But such regions are in the set S. However, for the sets S_1 defined by $\alpha_1-\epsilon\leqq\alpha\leqq\alpha_1$ and S_2 defined by $\alpha_2\leqq\alpha\leqq\alpha_2+\epsilon(\epsilon>0)$, in which $P(\alpha)$ is assumed to be increasing,

* We show below that condition (36) can be achieved.

we require that equation (36) holds in both S_1 and S_2 and *also* that

$$W(x_\alpha) < W(x_\beta)$$

where $\alpha \in S_1$ and $\beta \in S_2$. This last is true for the same reasons leading up to equation (36), namely, that in order to minimize $C-B$, we must reduce $W(x_\alpha)$ as α increases.

To demonstrate that equation (39) is also sufficient may be seen from equation (32). The first term merely gives B, which is independent of x_α and the second term depends only upon the *relative* size of the bribes and not upon the absolute bribe itself. However, equation (39) gives a complete description of the rank ordering of the bribes. Consequently the necessary and sufficient conditions for x_α to be an optimum bribing function is merely that it satisfy equation (31) and equation (39).

Thus the solution to the minimization problem set forth in equations (30) and (31) restricts x_α to be a strictly increasing function of α for $\alpha \notin S$. Having constrained only the mean bribe, we get only a condition on x_α rather than an explicit functional form; indeed, the solution is independent of the exact form of x_α, as long as it is strictly increasing with α. Thus, for the purposes of calculation and example we may choose some (simple) relation, such as the following linear one:

$$x_\alpha = K\alpha. \tag{40}$$

Applying the mean bribe constraint, we get

$$B = K\int_0^\infty \alpha \, dP(\alpha). \tag{41}$$

Letting A be the average impatience factor, we get from the last two equations,

$$x_\alpha = (B/A)\alpha. \tag{42}$$

This then is an optimal bribing function.

Considering the second case of waiting time $T(x_\alpha)$ in queue plus service facility, we note that

$$T(x_\alpha) = W(x_\alpha) + 1/\mu.$$

The solution here is therefore the same as above in Theorem 3 (*see also* equation 42).

For the preemptive priority rule the same results also apply as can be seen by noting that equations (1) and (18) are of the same form.

DISCUSSION AND EXAMPLE OF OPTIMUM BRIBING

IN ORDER TO obtain some insight into the behavior of the optimum bribing procedure and the cost function, we offer the following example. We consider a nonpreemptive system with an exponentially distributed bribe, viz.,

$$B(x)=1-e^{-\sigma x}. \qquad (\sigma \geqq 0) \quad (43)$$

As before, we have Poisson arrivals (mean rate λ) and an arbitrary service time distribution $F(\tau)$ (with mean $1/\mu$ sec). We assume (for purposes of normalization) that the first two moments of τ are chosen such that $W_0=\rho$,

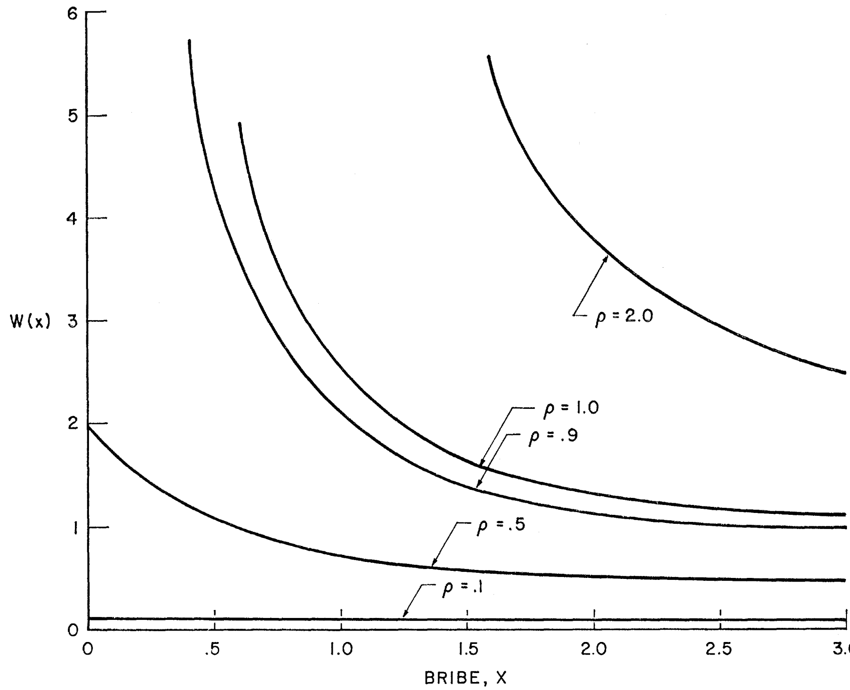

Fig. 1. Waiting time as a function of bribe.

where $\rho=\lambda/\mu$. Further, we choose to vary ρ by varying λ at a fixed μ, so that the input traffic is described by a single independent variable ρ.

For such a system, we may calculate $W(x)$ from equation (1) of Theorem 1. We plot the results of these calculations in Fig. 1 where we show $W(x)$ as a function of the bribe x with ρ as a parameter ($\rho=0.1$, 0.5, 0.9, 1.0, 2.0) and where we have taken $\sigma=1$ (the horizontal axis may be scaled by $1/\sigma$ to yield results for other values of σ). As expected and as shown in equation (38), the average wait is a decreasing function of the bribe x and an increasing function of the utilization factor ρ. Observe that under saturated conditions (i.e., $\rho \geqq 1$) we obtain finite average waits for all those customers offering bribes greater than x_{crit} where

$$x_{\text{crit}}=\begin{cases}0, & (\rho<1)\\ B^{-1}[(\rho-1)/\rho], & (\rho\geqq 1)\end{cases} \tag{44}$$

where $B^{-1}(z)$ is that value of x for which $B(x)=z$. This behavior is similar to that described in references 1 and 5 for the head-of-the-line priority system. We further note that, in general,

$$\lim_{x\to\infty} W(x)=W_0. \tag{45}$$

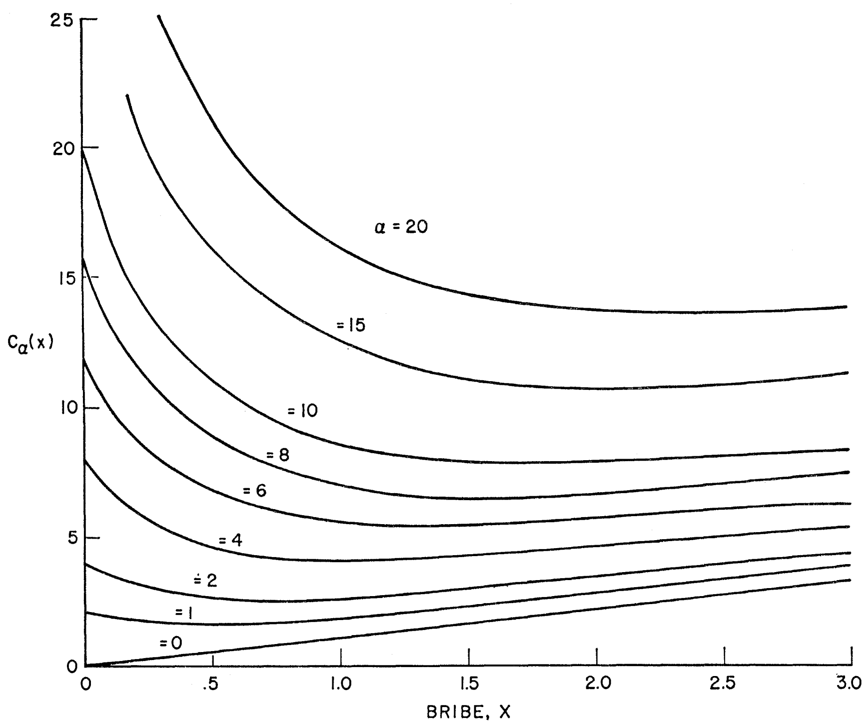

Fig. 2. Cost as a function of bribe at various impatience factors (ρ = 0.5).

For any given value of impatience factor, α, we can determine the cost $C(\alpha)$ as in equation (29), viz.,

$$C(\alpha)=x_\alpha+\alpha W(x_\alpha). \tag{29}$$

This cost is plotted in Fig. 2 with α as a parameter ($\alpha=0$, 1, 2, 4, 6, 8, 10, 15, 20) and for $\rho=0.5$. We plot $C(\alpha)$ versus x, taking the view that the bribe itself is the random variable. We observe in Fig. 2, the occurrence of a minimum of $C(\alpha)$ at some x for each fixed α.

The optimum bribing procedure for this example is obtained as follows. First, we alter our point of view and consider α to be the random variable with $x=x_\alpha$ a deterministic function of α. For example, let us take the form chosen in equation (42), viz.,

$$x_\alpha = (B/A)\alpha, \tag{42}$$

where B = average bribe,

A = average impatience factor.

Of course, the distribution on α is also exponential, in particular

$$P(\alpha) = 1 - e^{-(B/A)\sigma\alpha}. \tag{46}$$

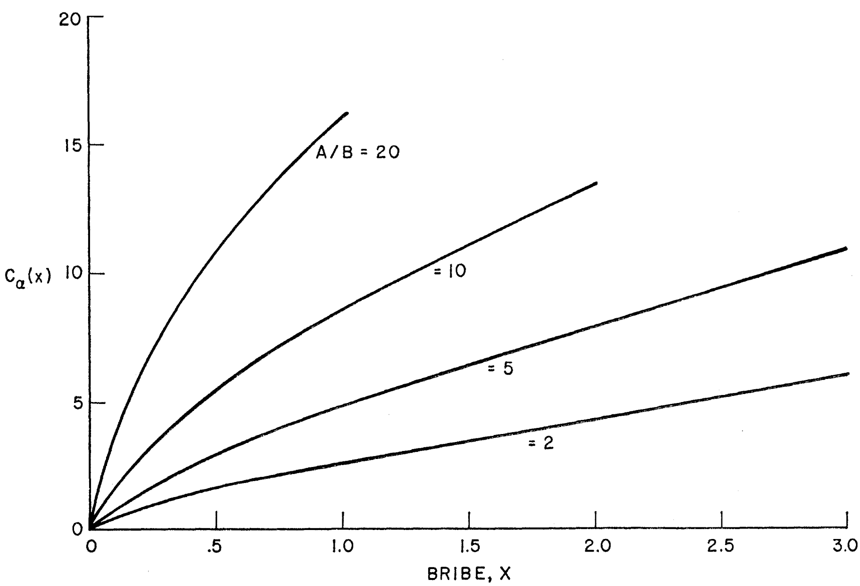

Fig. 3. Cost as a function of bribe for optimum bribing (ρ = 0.5).

Using the relation in equation (42) we can plot the cost $C(\alpha)$ versus x as shown in Fig. 3 with A/B as a parameter (recall A = average impatience factor, B = average bribe). Note that as the average impatience factor increases, so does the cost. As the average bribe, B, increases, it appears that the cost decreases, but of course this is only an illusion since we are plotting versus bribe x; thus as B varies, so does the weighting on the various values of x. In order to see the effect of changing B, we plot C versus the impatience factor α in Fig. 4. In this figure we again consider A/B as a parameter, with $\rho=0.5$. The curves in the approximate range $0 \leqq \alpha \leqq 20$ come from Fig. 2. For $a>20$, we observe the following:

$$\lim_{\alpha\to\infty} W(x_\alpha)=W_0.$$

Thus
$$C(\alpha)\sim[(B/A)+W_0]\alpha. \tag{47}$$

This asymptotic linear form for $C(\alpha)$ is shown in Fig. 4. The region between the measured values from Fig. 3 (for $\alpha<20$) and the asymptotic values given by equation (47) (shown as straight lines) has been filled in (by eye) as dashed lines.

Of further interest to this discussion is the value of the expected cost C as defined in equation (30), viz.,

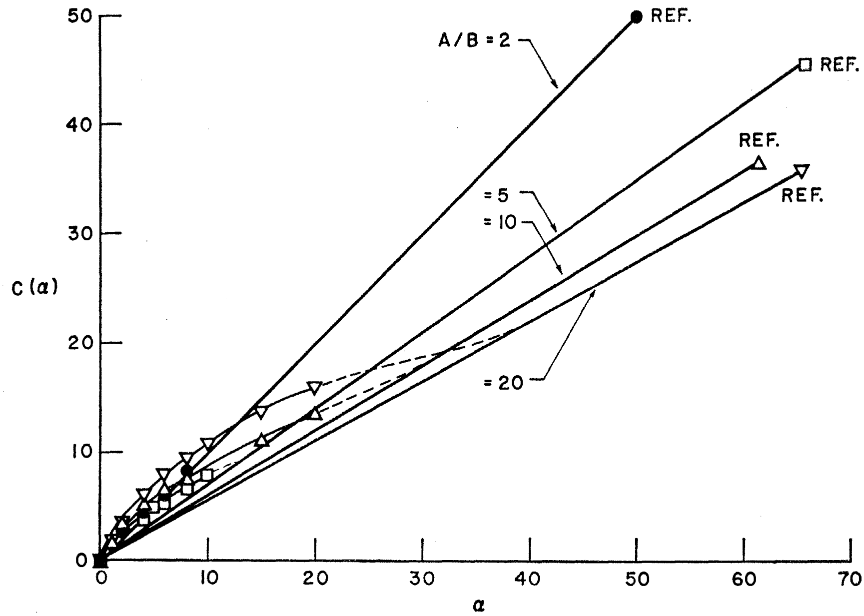

Fig. 4. Cost as a function of impatience factor for optimum bribing (ρ = 0.5).

$$C\equiv\int_0^\infty C(\alpha)\,dP(\alpha).$$

Applying equation (29) to this, we obtain [using equations (42) and (3) as well]:

$$C=B+\frac{A}{B}\int_0^\infty \frac{xW_0dB(x)}{[1-\rho+\rho B(x)]^2}. \tag{48}$$

Unfortunately, no more explicit expression for C has been found for general distributions $B(x)$. However, as examples, we calculate below the form that C takes for (i) exponentially and for (ii) uniformly distributed bribes: For $0 \leqq \rho < 1$, and

(i) For $dB(x)/dx = \sigma\epsilon^{-\sigma x}$, $C = (1/\sigma) + A\ln[1/(1-\rho)]$, (49)

(ii) For $dB(x)/dx = \begin{cases} 1/M & 0 \leqq x \leqq M, \\ 0 & \text{otherwise,} \end{cases}$ (50)

$$C = (M/2) - 2A + 2A(1/\rho)\ln[1/(1-\rho)].$$

It is interesting to note from equations (49) and (50) that the factor $\ln[1/(1-\rho)]$ appears in the average cost C; such a factor seems rather unusual in queuing problems.

CONCLUSION

In this paper, an analysis has been carried out that views priority queuing as a customer bribing mechanism. The average waiting times were calculated for Poisson input and arbitrary service distributions, both with and without preemption. The notion of an 'impatience' factor was then introduced, which allowed a cost function to be defined. We found that the only condition necessary for an optimum (in the sense of minimizing the average cost subject to a mean bribe constraint) bribing function was that it be monotonically increasing with the impatience factor. It was shown that many well-known priority disciplines may be viewed as bribing mechanisms.

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