Rude-CSMA: A Multihop Channel Access Protocol

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Abstract—**In this paper, we define a two-parameter family of protocols designed for multihop packet radio networks. We call these protocols *rude-CSMA* because under certain circumstances, maximum throughput is obtained when nodes, even after sensing a busy channel, transmit packets anyway with a nonzero rate. The performance of these protocols is analyzed for various special and random topologies.**

I. Introduction

Let us first review the operation of the CSMA protocol as defined for the single-hop environment. It is well known that CSMA [1] is an efficient channel access protocol for single-hop packet ratio environments. In this protocol, nodes of the network sense the channel prior to transmitting packets. If the channel is sensed busy, the sensing node refrains from transmitting (to avoid a collision) and reschedules its transmission according to one of several strategies until a future time. If the channel is sensed idle, the node transmits its packet. Collision of packets only occur if two or more nodes, after sensing an idle channel, start transmitting packets within a propagation time of each other.

In multihop networks, collisions can occur for yet another reason, namely, because of the hidden terminal problem [2]. Because of multihop, an idle channel around the transmitter does not necessarily imply that the channel around the intended receiver is also idle. As an example, consider the network of Fig. 1 where connectivity is shown by the arcs connecting the nodes. If node 1 senses an idle channel and sends a packet to node 2, the packet will suffer a collision if node 4, which is *hidden* from node 1 because it is outside of node 1's hearing range, is transmitting. This hidden terminal problem decreases the throughput that is achievable using CSMA in a multihop network.

Rude-CSMA attempts to gain back some of this lost throughput by transmitting sometimes *even if the channel is sensed busy*. The motivation is that a busy channel around the transmitter does not necessarily imply that the intended receiver of the packet also hears a busy channel. To see that this policy might improve the throughput of the system, again consider the case where node 1 has a packet for node 2 in Fig. 1. If node 1 senses a busy channel, it could be from node 3 which lies outside of the hearing range of node 2, and thus would not be responsible for causing a collision at node 2.

To create a mathematical model to analyze such a protocol, we will assume that when nodes sense the channel to determine if it is busy, they are given additional information. This information is the number of nodes and number of transmitting nodes in their local neighborhood. Clearly, this assumption is unrealistic, but this does not influence the main result of this paper which presents an important negative result. This conclusion is that the protocol that maximizes the throughput of the system ignores this information. The exceptions to this result occur for networks which are not likely to occur in actual network implementations.

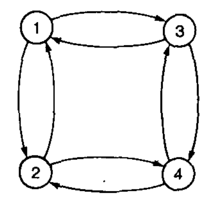

Fig. 1. A sample network.

The outline of our paper is as follows. In Section II, we describe the rude-CSMA protocol and derive equations for its performance. In Section III, we provide specific results for the networks we considered, and in Section IV we state our conclusions. The Appendix contains the mathematical details for the derivation of the equilibrium density for our protocol.

II. The Protocol

We will define the state of the network at time t, $S(t)$, to be a binary vector $S(t) = (s_1(t), s_2(t), \cdots, s_n(t))$ where n is the number of nodes in the network (which is assumed to be constant) and $s_i(t) = 1$ if node i is transmitting a packet at time t and 0 otherwise. Whenever node i is transmitting, ($s_i(t) = 1$), we assume that the packet transmission time is exponentially distributed with an average duration of $1/\mu$ time units, and thus that the rate at which node i's transmission is completed is given by

$$r_1{}^i(S(t)) = \mu. \tag{1}$$

When talking about equilibrium states, we will drop the t dependency in our notation and write the state and its components as S and s_i. We also define $\bar{s}_i = 1 - s_i$.

For any node i, there exists a subset of the other nodes of the network, denoted by A_i, which it can hear. These nodes will be referred to as being neighbors of node i. In this paper, we will assume that if node i can hear node j, then node j can also hear node i, and thus if $i \in A_j$, then $j \in A_i$. We let $|A_i|$ be the number of elements of A_i. In general, for a multihop network, A_i is a proper subset of all the nodes in the network. Thus, the information that i uses to determine when it transmits depends only upon the states of the nodes contained in A_i. For a given state $S(t)$, suppose that node i has a packet which is ready to be transmitted. The number of nodes that are both neighbors of i and transmitting is given by

$$N_1{}^i(S(t)) = \sum_{j \in A_i} s_j(t).$$

Likewise, the number of neighbors not transmitting in i's neighborhood is given by

$$N_0{}^i(S(t)) = |A_i| - N_1{}^i(S(t)).$$

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We assume that $N_1{}^i(S(t))$ and $N_0{}^i(S(t))$ are given to a node whenever it senses the channel. According to these values, node i adjusts the rate at which it presents packets to the channel. For state $S(t)$, using rude-CSMA, the rate at which node i transmits (assuming that $s_i(t) = 0$) is given by

$$r_0{}^i(S(t)) = \gamma_0 x^{N_0{}^i(S(t))} y^{N_1{}^i(S(t))} \tag{2}$$

where x, y, and γ_0 are given network parameters. Thus, for state $S(t)$, the time between successive packet transmissions by node i is exponentially distributed with mean $1/r_0{}^i(S(t))$. We can interpret (2) in the following manner: γ_0 corresponds to the *nominal arrival rate* of new, relayed, and retransmitted packets at each node of the network. If the local state information had no influence on node i's behavior (in terms of network parameters, this corresponds to $x = 1, y = 1$), then γ_0 would represent the rate at which nodes transmit packets on the channel, independent of the state of the system. This would be identical to the offered channel load if one were using the ALOHA protocol. The influence of the local environment on node i's behavior is represented by the values x and y. Note that ALOHA ($x = 1, y = 1$) and CSMA ($x = 1, y = 0$) appear as special cases so that rude-CSMA is a generalization of these two well-known protocols.

The state vector and rates defined above define a continuous time, finite multidimensional state Markov process. A generalized form of this process can be found in [3]. To optimize the performance of the network, we need to determine the (x, y) values that maximize the expected number of successful transmissions over a unit time interval. For a given equilibrium state S, define $U(S)$ to be the expected number of concurrent successful transmissions for that state. The value $U(S)$ can be calculated if the traffic and hearing matrix for the system are known. As an example, suppose in Fig. 2 that nodes are equally likely to transmit to any of their neighbors. Nodes in Fig. 2 are labeled (1) if they are transmitters and (0) otherwise. The value of $U(S)$ is found by calculating the probability that silent nodes in the network successfully receive a packet addressed to them. Thus, node 2 in the figure is adjacent to only one transmitting node (node 1) which transmits in that direction with probability 1/4. The probability of a successful transmission to node 2 is 1/4, as it is for nodes 3 and 4. In the same manner, nodes 6 and 7 have a 1/4 probability of receiving node 5's transmission successfully, whereas node 8 has no chance of receiving either node 5's or node 9's reception since they will collide. For the state shown then, the expected success is

$$U(S) = 3\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) = \frac{5}{4}.$$

More generally, let p_{ji} be the probability that when node j transmits a packet, it sends it to node i. We then calculate $U(S)$ as

$$U(S) = \sum_{i=1}^{n} \bar{s}_i \delta_{N_1{}^i(S)} \sum_{j \in A_i} p_{ji} s_j$$

where δ_x equals 1 if $x = 1$ and equals 0 otherwise. Observe that $U(S)$ measures the expected number of concurrent transmissions that are received successfully when the system is in state S. In an actual packet radio network, this is equivalent to assuming that nodes send a series of very short packets when they transmit. The use of this mechanism to calculate throughput was first presented in [4]. We should also point out that a model with similar assumptions was studied in [5] in which the mathematical model corresponded to a reversible Markov process that had a product form solution. In that work, CSMA was the protocol studied.

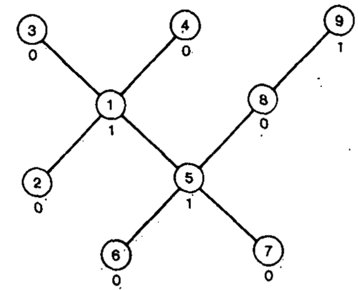

Fig. 2. Calculating $U(S)$.

Let $\Pi(S, x, y)$ be the steady-state probability of state S for a given (x, y) and denote the set of all possible states as Ω. In the Appendix, we show that with rate equations (1) and (2), $\Pi(S, x, y)$ is given by

$$\Pi(S, x, y) = C\rho^{M(S)} x^{-B_0(S)} y^{B_1(S)}$$

where

$$M(S) = \sum_{i=1}^{n} s_i$$

$$C = \Pi(0, x, y) x^{-B_0(0)}$$

$$\rho = \gamma_0/\mu$$

$$\Pi(0, x, y) = \left[x^{-B_0(0)} \sum_{S \in \Omega} \rho^{M(S)} x^{-B_0(S)} y^{B_1(S)} \right]^{-1}$$

In this equation, C is a normalization constant, $M(S)$ is the number of transmitters in the network, $B_0(S)$ is the number of pairs of nodes in the network that are both adjacent to (i.e., can hear) each other and are not transmitting, and $B_1(S)$ is the number of pairs of nodes in the network that are both adjacent to each other and also are both transmitting. Using this, we can write the expected number of successful transmissions in the network as

$$U(x, y) = \sum_{S \in \Omega} \Pi(S, x, y) U(S).$$

Naturally, we would like to maximize this function over feasible (x, y) values. Besides the nonnegativity of x and y, however, there is a constraint concerning the average rate at which a node presents packets to the channel. This average rate cannot be greater than the nominal packet arrival rate γ_0. Rate equation (2) shows that the actual transmission rate is a function of S, and thus we must have the following *flow constraint*:

$$\sum_{S \in \Omega} r_0{}^i(S) \bar{s}_i \Pi(S, x, y) \leqslant \gamma_0 \qquad i = 1, 2, \cdots, n.$$

For a given topology and traffic matrix, we can thus formalize the mathematical program P as

Program P

$$\max_{(x,y)} \; U(x, y) = \sum_{S \in \Omega} \Pi(S, x, y) U(S)$$

subject to

$$1 - \sum_{S \in \Omega} x^{N_0{}^i(S)} y^{N_1{}^i(S)} \bar{s}_i \Pi(S, x, y) \geqslant 0 \qquad i = 1, 2, \cdots, n$$

$$x \geqslant 0$$

$$y \geqslant 0.$$

This is the problem that we will address in this paper. We note that optimal (x, y) values are functions of the topology and traffic matrix of the network. To solve program P, we used an exhaustive grid search method since convexity properties of the objective function were found to be difficult to establish, and they also depend in each case on the topology and traffic matrix. For the examples that follow, we will denote the optimal solution to program P by $U^*(\rho)$ where $\rho = \gamma_0/\mu$.

III. Discussion of Results

In this section, we will discuss the results we obtained when P is optimized for example networks. We wrote computer programs which, for a given topology, calculated $U(S)$ and $\Pi(S, x, y)$ and the optimized P over all feasible (x, y) pairs. Besides special topologies which we studied, we also ran P for connected networks which were randomly generated for varying mean densities. As typical of most Markovian models, the state space of the system grows exponentially with the number of nodes of the network, and thus to keep the program computationally tractable, we restricted our optimization to relatively small networks. We have chosen here to show results obtained for a six-node grid network and for a network that was randomly generated. The results for these networks are representative of our studies. In both networks, we have assumed a uniform local traffic matrix. Results for other topologies, such as rings and tandems, and for other random networks can be found in [6].

A. A Six-Node Grid Network

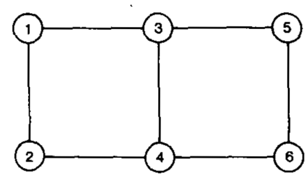

Fig. 3. A six-node grid network.

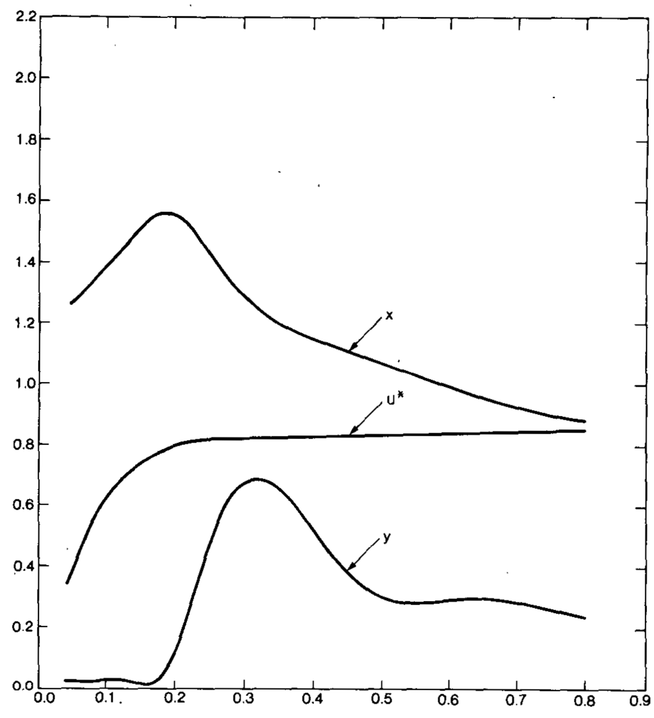

Fig. 4. Curves for the six-node grid network.

In Fig. 4 we plot the values of x and y that achieve optimal performance for the network of Fig. 3, which is a generalized version of the network used to motivate this work. The curves for the graph in Fig. 1 show similar behavior. These curves have many interesting properties. First we observe that over the range of $\rho = \gamma_0/\mu$ shown, there are three distinct types of behavior. For very small ρ values, $\rho \leqslant 0.15$, we see that x and $U^*(\rho)$ increase rapidly, while y remains very small. The increase of $U^*(\rho)$ over this range is explained by the fact that for small ρ values, there are very few transmissions, and thus very few which cause collisions, and thus increasing ρ tends to increase $U^*(\rho)$ in a linear manner. We see this behavior clearly as ρ goes from 0.05 to 0.1 during which $U^*(\rho)$ doubles from about 0.35 to about 0.6. As ρ continues to grow beyond 0.1, the increase is less than linear due to some collisions in the network. It is clear that over this range, since there are so few transmissions in the network, the y parameter of rate equation (2) is noncritical. Using an interactive program we wrote to determine how $U^*(\rho)$ varied as a function of y, we found very little change over all values of y; thus, the sudden increase in y at $\rho \approx 0.2$ should not be interpreted as demonstrating singular behavior.

The second region of the graph, from $\rho = 0.2$ to $\rho = 0.35$, will now be explained. First, we observe that the expected success rises only slightly over this range. Although there are more transmissions in this range, there are also more collisions which limit the number that are successful. The sudden increase of y at $\rho \approx 0.2$, as mentioned previously, should not be looked on as being as singular, but does demonstrate the increased importance of y's effect on the throughput of the system. Observe that over this range $y < 1$, and thus from (2), this parameter acts to inhibit transmissions when there is a neighboring transmitter (although not to the extent of excluding such transmissions). This supports our conclusion from the scenario in the beginning that sometimes in a multihop network, it is beneficial to transmit even if the channel is sensed busy. The decrease of x over this range arises because it is no longer on the boundary of the feasible region. However, since x is greater than 1 in this region, we conclude that this parameter tends to increase the rate of packets offered to the channel during idle channel periods.

Over the region $\rho \geqslant 0.35$, $U^*(\rho)$ is again not altered to any great measure. Both x and y decrease now in an effort to prevent collisions on the channel. As ρ becomes much larger, we see that $x < 1$, which indicates that even the x parameter tends to act to inhibit the number of packets on the channel. Over this region, the flow constraint of P is not saturated for any node in the network, indicating that the effective transmission rate of packets on the channel is less than γ_0. Hence, optimizing the throughput of the network has the effect of limiting the flow of packets in the network. We will explain the general shape of the x curve in the next section.

B. A Seven-Node Random Network

We now report on the results which were obtained when P was run on graphs which were randomly generated. Many such graphs were created, with varying mean densities, and all such graphs exhibited similar behavior. This characteristic leads us to conclude that in a random multihop network, over the continuum of protocols defined by all possible (x, y) pairs, optimum performance is obtained when $y = 0$. As stated in the Introduction, this corresponds to a CSMA-type protocol. From the many networks which were analyzed, we have chosen one to show here. In Fig. 5, we show a seven-node network with a mean number of neighbors N equal to 2.28. The corresponding set of curves are shown in Fig. 6. We see that for low values of ρ, both $U^*(\rho)$ and x are linear, indicating that there are very few collisions in the network. The slope of $U^*(\rho)$ becomes nearly zero for $\rho > 0.2$, implying that after this point, the amount of new successful transmissions in the network is small in comparison to the number of collisions that arise

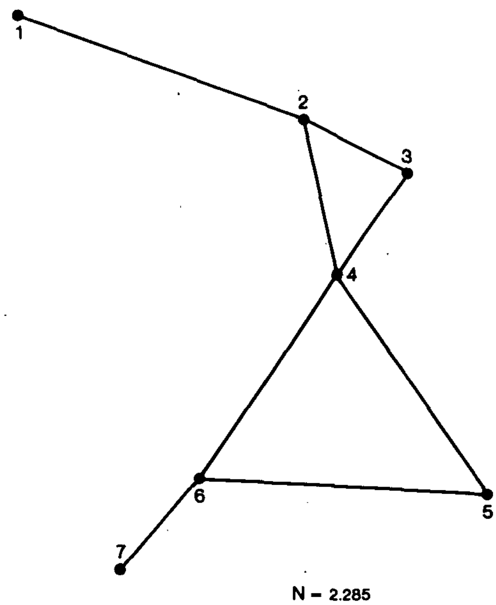

Fig. 5. A random seven-node network with $N = 2.285$.

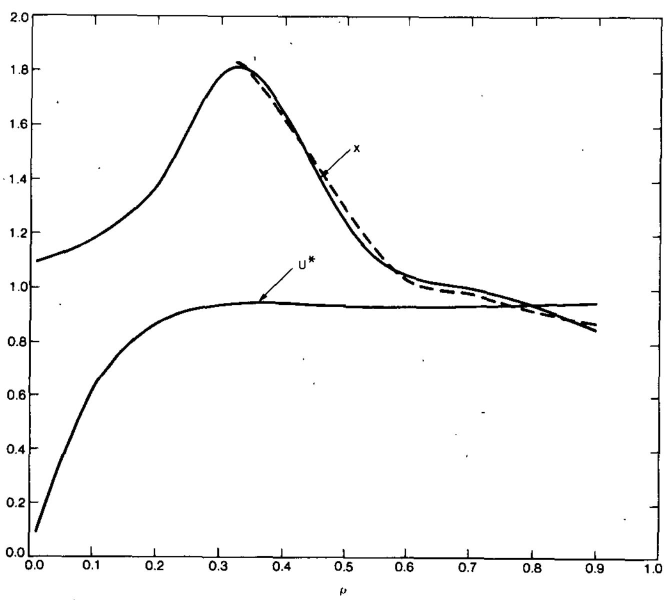

Fig. 6. Curves for the seven-node random network.

from the increased traffic load. The shape of the x curve for $\rho > 0.3$ can be explained by first noting that over this range, the flow constraint of P is not saturated. Next we make a definition. Suppose for each node i, we investigate all state vectors having the property that i and all its neighbors A_i are silent. Under these conditions, if i senses the channel, it will detect that it is idle and offer packets to the channel at a rate defined by the protocol. Let I_i be the set of all state vectors satisfying this condition. We can then define the *effective number of idle neighbors,* denoted by N^*, by finding the average number of silent neighbors a randomly selected idle node has when it senses the channel prior to a transmission. This can be written as

$$N^* = \sum_i |A_i| \sum_{S \in I_i} \Pi(S, x, y).$$

Because of topological variations, some nodes are more likely to be both idle and surrounded by idle neighbors than others (in fact, the probability is decreasing with the number of neighbors), and thus in general, $N^* \neq N$. Define f to be the average time a random node in the network waits before transmitting when it hears an idle channel. Since $U^*(\rho)$ does not vary much over $\rho > 0.3$, it is not surprising to find that x varies as a function of ρ to preserve the fraction of time nodes transmit when sensing an idle channel. This has been verified by observing (using an interactive program) that the probability distribution for state vectors does alter significantly as ρ ranges greater than 0.3 for optimized x values. We thus can equate rates to obtain

$$\rho x^{N^*} = 1/f.$$

We can solve for f and N^* by choosing two points from Fig. 6. At $\rho = 0.72$, we have $x = 1$, and thus can write $0.72 = 1/f$ or $f = 1.36$. At $\rho = 0.32$, we have $0.32\ (1.8)^{N^*} = 0.72$, which implies that $N^* = 1.4$. Thus, we can write x as a function of ρ as

$$x = (1/f\rho)^{1/N^*} = (0.72/\rho)^{0.71} \qquad \rho > 0.3. \tag{3}$$

Equation (3) was used to generate the second curve in Fig. 6 and we see a relatively close match. In general, then, for a CSMA environment, we can write a general equation relating x with ρ as

$$x = (1/f\rho)^{1/N^*}.$$

Intuitively, for the graph of Fig. 5, we would expect $N^* < N$ since nodes having the greater number of neighbors, node 4 for instance, have a much smaller probability of being both idle and surrounded by idle neighbors than a node, say node 1, that has far fewer neighbors.

IV. CONCLUSIONS

In this paper, we have defined rude-CSMA which is a two-parameter family of protocols for multihop channel access. This family of protocols has been shown to include both ALOHA and CSMA as special cases. The optimum performance of rude-CSMA was presented for two sample networks. In the six-node grid network, we showed that in the optimal solution, nodes transmit with a nonzero rate even if they sense a busy channel. This explains the appelation *rude* that we have given the protocol. In the more realistic case of random networks, however, optimal performance was obtained when the protocol used was CSMA with optimized channel input rates. Thus, we must come to the interesting conclusion that for practical networks, rude-CSMA is not that *rude* after all.

APPENDIX

In this Appendix, we will determine a formula for $\Pi(S, x, y)$, and this will permit us to explain the particular definition that was needed in the main body of the paper for the rates defined in (1) and (2). It turns out that with these rate definitions, determining the steady-state probability distribution for any given topology is mathematically tractable because the Markovian process defined by these rates defines a *reversible process* [7]. Intuitively, a reversible process $X(t)$ is one in which the direction of time has no effect on the statistics of the process, and thus $X(t)$ and $X(-t)$ have the same probability distribution. Finding closed form solutions for reversible processes is simplified because reversible processes satisfy detailed balanced equations. Let $\Pi(S) = \Pi(S, x, y)$ be the steady-state probability of state S, and let $q(S, S')$ (we will shortly define this for our system) be the rate at which transitions between states S and S' occur. The detailed state balance equations are

$$\Pi(S)q(S, S') = \Pi(S')q(S', S) \qquad \forall S, S'. \tag{A.1}$$

In general, Markov processes satisfy global balance equations that equate the total probability flux entering a state to that leaving the state. These equations state that

$$\sum_{S'} \Pi(S')q(S', S) = \sum_{S} \Pi(S)q(S, S'). \tag{A.2}$$

We can depict (A.2) in Fig. 7 where the equation states that the net probability flux across the cut is zero. For reversible systems, (A.1) shows that the net flow across arcs connecting any two states (the cut in Fig. 8) is equal to zero. Equation (A.1) can be used to simplify finding closed form solutions if reversibility can be proven, and a useful tool to do this is *Kolmogorov's criteria.*

Theorem (Kolmogorov's Criteria): Let S^1, S^2, $\cdots$, S^k, $S^{k+1} = S^1$ be any cycle of states. Then a Markov process defined over these states is reversible if and only if the rate transitions satisfy

$$q(S^1, S^2)q(S^2, S^3) \cdots q(S^{k-1}, S^k)q(S^k, S^{k+1})$$
$$= q(S^{k+1}, S^k)q(S^k, S^{k-1}) \cdots q(S^3, S^2)q(S^2, S^1).$$

To determine $\Pi(S, x, y)$ for our system, we will first need to make some preliminary definitions. Let T_i be an operator acting on state S which complements the ith component; thus,

$$T_i(S) = (s_1, s_2, \cdots, s_{i-1}, \bar{s}_i, s_{i+1}, \cdots, s_n)$$

and we will say a transition is a $0 \rightarrow 1$ transition if $s_i = 0$, and for some j, we have $T_j(S) = S'$ ($1 \rightarrow 0$ transitions are defined in the analogous way). Using this, we can define the transition rates for our system as

$$q(S, S') = \begin{cases} 0 & T_i(S) \neq S' \text{ for all } i \\ \gamma_0 x^{N_0{}^i(S)} y^{N_1{}^i(S)} & T_i(S) = S' \text{ and } s_i = 0 \\ \mu & T_i(S) = S' \text{ and } s_i = 1. \end{cases}$$

We can now prove a lemma that will be used to show that the process defined by rate equations (1) and (2) is reversible. To avoid any possible confusion, we should point out that $|X|$ denotes the number of elements contained in X if X is a set, and is the absolute value of X is X is a real number.

Lemma: For any state S, $N_i{}^j(S) = |B_i(S) - B_i(T_j(S))|$.

Proof: For a given j,

$$B_i(S) = |\{\{j, k\} | k \in A_j, s_j = s_k = i\}|$$
$$+ |\{\{m, n\} | m, n \neq j, m \in A_n, s_m = s_n = i\}| \tag{A.3a}$$
$$B_i(T_j(S)) = |\{\{j, k\} | k \in A_j, \bar{s}_j = s_k = i\}|$$
$$+ |\{\{m, n\} | m, n \neq j, m \in A_n, s_m = s_n = i\}|. \tag{A.3b}$$

The second summands in each of (A.3a) and (A.3b) are identical and thus cancel in $|B_i(S) - B_i(T_j(S))|$. Only one of the first summands can be nonzero since either $s_j = i$ or $\bar{s}_j = i$, and thus

$$|B_i(S) - B_i(S)T_j(S)|$$
$$= |\{\{j, k\} | k \in A_j, s_j = s_k = i\}$$
$$- \{\{j, k\} | k \in A_j, \bar{s}_j = s_k = i\}|.$$

Since we are taking the absolute value of the above expression, its value does not change if we assume that $s_j = i$ and write

$$|B_i(S) - B_i(T_j(S))|$$
$$= |\{\{j, k\} | k \in A_j, s_k = i\}| = |\{k \in A_j | s_k = i\}| = N_i{}^j(S).$$

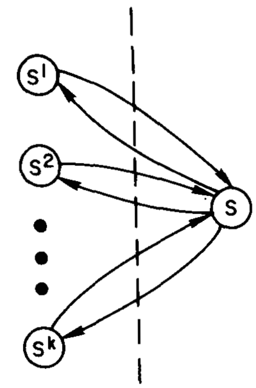

Fig. 7.

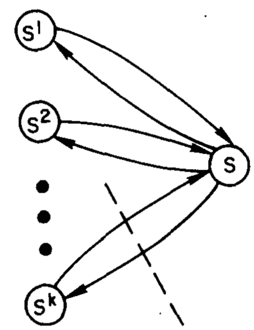

Fig. 8.

Observe that for a known $0 \rightarrow 1$ transition involving node j, we can eliminate the absolute value sign in the above to get $N_0{}^j(S) = B_0(S) - B_0 T_j(S)$. In a like manner, if node j is involved in a $1 \rightarrow 0$ transition, we have $N_1{}^j(S) = B_1(T_j(S)) - B_1(S)$. We are now in a position to prove that the Markov process described before is reversible.

Theorem: The Markov process described by rate definitions (1) and (2) defines a reversible Markov process.

Proof: We will show that Kolmogorov's criteria is satisfied. For any cycle of states $S^1, S^2, \cdots, S^k, S^{k-1} = S^1$, Kolmoorov's criteria are trivially satisfied if $S^{i+1} \neq T_j(S^i)$, $i = 1, 2, \cdots, k$ for all j since the probability of two or more components of S^i and S^{i+1} differing is zero. This follows from the fact that the process is continuous in time. Thus, assume that the cycle of states consists of single component transitions. Since there are only two types of transitions, all cycles must contain an even number of states. Let $S^1, S^2, \cdots, S^{2m}, S^{2m+1} = S^1$ be such a sequence. Let

$$\Psi_1 = q(S^1, S^2)q(S^2, S^3) \cdots q(S^{2m}, S^{2m+1})$$
$$\Psi_2 = q(S^{2m+1}, S^{2m}) \cdots q(S^3, S^2)q(S^2, S^1).$$

We must show that $\Psi_1 = \Psi_2$, and we will refer to their corresponding sequences as the forward and backward sequence, respectively. We first note that there is an equal number of $0 \rightarrow 1$ and $1 \rightarrow 0$ transitions in Ψ_1 and Ψ_2 since the state sequence is circular. Using the rate equations $q(S, S')$, we can write $\Psi_i = \mu^m \gamma_0{}^m \Phi_i$ where Φ_i contains all the factors of x and y. We thus must show that $\Phi_1 = \Phi_2$. We will first concentrate on the exponent of x in these equations. We will first make an observation about the two sequences Ψ_1 and Ψ_2. Since they are both cyclic and reversals of each other, a $0 \rightarrow 1$ transition for $S^i \rightarrow S^{i+1}$ in Ψ_1 corresponds to a $1 \rightarrow 0$ transition $S^{i+1} \rightarrow S^i$ in Ψ_2. Define the two sets:

$$C_1 = \{(S^i, S^{i+1})$$
$$i = 1, 2, \cdots, 2m \mid \exists j T_j(S^i) = S^{i+1}, s_j{}^i = 0\}$$

$$C_2 = \{(S^{i+1}, S^i) \quad i = 1, 2, \cdots, 2m \mid \exists j T_j(S^{i+1}) = S^i, s_j^{i+1} = 0)\}$$

where s_j^i is the jth component of state S^i. In words, C_1 contains the $0 \to 1$ transitions for the forward sequence and C_2 contains the $0 \to 1$ transitions for the backward sequence. The x exponent will only change due to these $0 \to 1$ transitions. Since C_1 only contains $0 \to 1$ transitions, we can write, using the lemma, the exponent of x in the forward E_1 and backward E_2 sequences as

$$E_i = \sum_{(S,S') \in C_i} B_0(S) - B_0(S').$$

Suppose now that $S^i, S^{i+1}, \cdots, S^{i+k}$ is a subsequence of states from the forward sequence such that $(S^j, S^{j+1}) \in C_1$, $j = i, i+1, \cdots, i+k-1$. Since the portion of E_1 for this sequence alternates sign, the sum telescopes, and we can write for this subsection

$$\sum_{j=i}^{j=i+k-1} B_0(S^j) - B_0(S^{j+1}) = B_0(S^i) - B_0(S^{i+k}).$$

With this in mind, we see that to calculate E_1, we only have to look at sections in the forward sequence where a change in the type ($0 \to 1$ to $1 \to 0$) of state transition occurs. Thus, let us define $S'^i, i = 1, 2, \cdots, 2k, 2k+1$ with $S'^{2k+1} = S'^1$ to be the places where such changes in the types of transitions occur. To be precise, if $S'^{2n-1} = S^i$ and $S'^{2n} = S^{i+k}$, then $(S^{i-1}, S^i) \notin C_1$, $(S^j, S^{j+1}) \in C_1$, $j = i, i+1, \cdots, i+k-1$, and $(S^{i+k}, S^{i+k+1}) \notin C_1$. Using this, we can then write E_i as

$$E_1 = [B_0(S'^1) - B_0(S'^2)] + [B_0(S'^3) - B_0(S'^4)] + \cdots + [B_0(S'^{2k-1}) - B_0(S'^{2k})].$$

In a like manner, we can write E_2 as

$$E_2 = [B_0(S'^1) - B_0(S'^{2k})] + [B_0(S'^{2k-1}) - B_0(S'^{2k-2})] + \cdots + [B_0(S'^3) - B_0(S'^2)].$$

Rearranging these sums shows that $E_1 = E_2$.

The proof that the y exponents of Φ_1 and Φ_2 are identical follows from a similar argument. One would define analogous sets to C_1 and C_2 above that had all the $1 \to 0$ transitions and proceed in an identical fashion. This concludes the proof that the process is reversible.

Knowing that the Markov process is reversible allows us to use the detailed balance equations (A.1) to prove the following theorem.

Theorem A.1: The equilibrium probability for state S with transition rates defined by (1) and (2) is given by

$$\Pi(S, x, y) = C\rho^{M(S)} x^{-B_0(S)} y^{B_1(S)} \tag{A.4}$$

where

$$M(S) = \sum_{i=1}^{n} s_i$$

$$C = \Pi(0, x, y) x^{-B_0(0)}$$

$$\rho = \gamma_0/\mu$$

$$\Pi(0, x, y) = \left[x^{-B_0(0)} \sum_S \rho^{M(S)} x^{-B_0(S)} y^{B_1(S)} \right]^{-1}.$$

Proof: To avoid cumbersome notation, denote $\Pi(S, x, y)$ by $\Pi(S)$. Since the process is reversible, we can use the detailed balance equations to state

$$\frac{\Pi(S)}{\Pi(T_j(S))} = \frac{q(T_j(S), S)}{q(S, T_j(S))}. \tag{A.5}$$

If $s_j = 1$, we can write (A.5) as

$$\frac{\Pi(S)}{\Pi(T_j(S))} = \rho x^{N_0{}^j(S)} y^{N_1{}^j(S)}$$

which, by using the lemma, can be rewritten (for the case of a $1 \to 0$ transition) as

$$\frac{\Pi(S)}{\Pi(T_j(S))} = \rho x^{[B_0(T_j(S)) - B_0(S)]} y^{[B_1(S) - B_1(T_j(S))]} \qquad s_j = 1. \tag{A.6}$$

We can use this relationship to write $\Pi(S)$ in terms of $\Pi(0)$ by telescoping a product of rate ratios. Suppose $i_1, i_2, i_3, \cdots, i_{M(S)}$ are the indexes of S which are equal to 1. Define the following state operator:

$$F_0 = 1$$

$$F_j = T_{i_j} F_{j-1} \qquad j = 1, 2, \cdots, M(S).$$

Observe that $F_0(S) = S$ and $F_{M(S)} = 0$ (a vector of all zeros). We can then write (A.5) as

$$\frac{\Pi(S)}{\Pi(0)} = \frac{\Pi(F_0(S))}{\Pi(F_1(S))} \frac{\Pi(F_1(S))}{\Pi(F_2(S))} \cdots \frac{\Pi(F_{M(S)-1}(S))}{\Pi(F_{M(S)}(S))}$$

$$= \frac{q(F_1(S), F_0(S))}{q(F_0(S), F_1(S))} \frac{q(F_2(S), F_1(S))}{q(F_1(S), F_2(S))} \cdots \cdot \frac{q(F_{M(S)}(S), F_{M(S)-1}(S))}{q(F_{M(S)-1}(S), F_{M(S)}(S))}$$

which can be simplified by using (A.6) to

$$\frac{\Pi(S)}{\Pi(0)} = \rho^{M(S)} x^{\sum_{i=0}^{M(S)-1} B_0(F_{i+1}(S)) - B_0(F_i(S))} \cdot y^{\sum_{i=0}^{M(S)-1} B_1(F_i(S)) - B_1(F_{i+1}(S))}$$

These sums telescope, and we are left with

$$\frac{\Pi(S)}{\Pi(0)} = \rho^{M(S)} x^{[B_0(F_{M(S)}(S)) - B_0(F_0(S))]} \cdot y^{[B_1(F_0(S)) - B_1(F_{M(S)}(S))]}.$$

We thus write

$$\Pi(S) = \Pi(0) \rho^{M(S)} x^{[B_0(0) - B_0(S)]} y^{[B_1(S) - B_1(0)]}$$

and using the fact that $B_1(0) = 0$ and defining $C = \Pi(0) x^{-B_0(0)}$, we finally obtain

$$\Pi(S) = C\rho^{M(S)} x^{-B_0(S)} y^{B_1(S)}. \tag{A.7}$$

We can determine $\Pi(0)$ by using the normalization constraint to determine the equation for $\Pi(0)$ given in the statement of Theorem A.1.

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