On the Capacity of Single-Hop Slotted ALOHA Networks for Various Traffic Matrices and Transmission Strategies

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Abstract—**In this paper we formulate a general model of the capacity of single-hop slotted ALOHA networks. We find that the capacity can be expressed as a function of the nodal degree (i.e., number of nodes within range of a transmitter). We than evaluate this model for various traffic matrices. In order to satisfy the requirements of a given traffic matrix, the transmission power is selected accordingly and this determines the degree of the nodes and, hence, the network performance. Finally we compare our results to simulation studies.**

I. INTRODUCTION

THE first use of a random access protocol for computer communication was the ALOHA system at the University of Hawaii, which was used to connect the various campuses of the University located on different islands via radio packet switching [1], [8]. In the ALOHA protocol, any node wishing to transmit over the (broadcast) channel does so with no regard for the other users. In light traffic the transmission will succeed with high probability, resulting in a low delay (compared to TDMA, for example). As the traffic load increases, the probability of a node deciding to transmit during the transmission of another node will also increase. This will result in a "collision" causing the destruction of one or both of the messages. The nodes involved in the collision are alerted to this fact by the subsequent lack of positive acknowledgment. They then retransmit the message after some (random) delay (to avoid continued collisions).

The early work in this area considered centralized fully connected networks (i.e., all nodes could hear one another and all traffic was directed to a particular central node such as the main campus in the ALOHA system). Extensive analysis of "pure ALOHA" for fully connected networks can be found in [1], [10], where it is determined that the maximum that the channel can be utilized is 18 percent ($1/2e$) of the channel bandwidth. A simple modification to the ALOHA scheme slotted ALOHA, proposed in [12]—forces transmissions to commence at the beginning of "slots" (time divisions of length equal to a packet transmission time). In [10], [12], we find analysis of this scheme showing that the capacity is doubled (over unslotted ALOHA), to 36 percent ($1/e$).

More recently, the concept of a packet radio network has been developed [5], [6], which allowed multihop communication but initially required all traffic to pass through central nodes called stations. Modifications to the original design also allow direct terminal-to-terminal communication without requiring the message to pass through the station.

In this paper we consider such direct terminal-to-terminal traffic but restrict our models to the case where the destination can be reached without the use of repeaters (single-hop). For some traffic patterns it is not necessary to use a fully connected topology, however. In fact, we shall see that it is beneficial to use reduced transmission power and reduce the interference.

We first develop a general model for the performance of one-hop slotted ALOHA broadcast networks that satisfy certain homogeneity assumptions and then evaluate the model for various combinations of traffic patterns and transmission strategies.

II. NETWORK MODEL

The network environment that we consider is a set of randomly located nodes which are able to communicate in one hop. Such a network may be thought of as a representative of all possible configurations or as a random snapshot of a mobile network. The traffic model is of the (instantaneous) communication requirement between some active subset of the total number of nodes in the network (nonactive nodes are ignored).

We consider a set of n (active) nodes located randomly according to a uniform distribution over a unit hypersphere in a space of some dimensionality. We consider that the nodes communicate in (symmetrical) pairs. If we let $T = t_{ij}$ represent the traffic rate from node i to node j, then $t_{ij} = 0 \Rightarrow t_{ij} = 0$, and T has exactly one nonzero element in each row and column. (Note that we do not require that $t_{ij} = t_{ji}$.)

For a particular network and traffic matrix, we satisfy the communication requirement by a suitable choice of transmission power. There are two approaches to satisfying this random communication pairing: 1) give every node sufficient power to be able to reach every other node in the network; or 2) give each node sufficient power to just reach his communication partner.

Once the network is established, as above, we have one additional parameter to specify—the probability that a node

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will transmit in any slot. (This corresponds to the offered channel traffic randomized so that slotted ALOHA will operate correctly and resolve previous conflicts due to simultaneous transmissions.) In order to compute the throughput, we use the "heavy traffic model," which corresponds to assuming that all (active) nodes are always busy, but which transmit independently in any given slot depending on this transmission probability. We denote the transmission probability for node i as p_i.

A. Nodal Throughput

Consider an arbitrary node (say node i) in the network. The probability that this node correctly transmits a packet to his partner (say node j) in any slot is given by

$$s_i = \Pr\{i \text{ transmits}\} \Pr\{j \text{ does not transmit}\}$$

$$\Pr\{\text{none of } j\text{'s neighbors transmits}\}$$

$$= p_i(1-p_j) \prod_{k \in N_j} (1-p_k) \tag{1}$$

where N_j is the set of nodes that j can hear (excluding his partner i). The assumption here is that reception is a discrete process, i.e., a node either hears a transmission or does not. Thus, another transmission either causes interference or not, depending on whether he is more distant than the threshold of reception. In a real network this reception process is not discrete but depends on relative power levels, noise, multipath, etc.

s_i is the rate at which node i succeeds in sending traffic to his communication partner. Since we have a *single-hop* network, this corresponds to the throughput for this node, γ_i. If the network were multihop we would have to divide this success rate by the path length in order to obtain end-to-end throughput. Thus, the total network throughput γ is given by

$$\gamma = \sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} s_i. \tag{2}$$

Since the networks we consider the homogeneous, the throughput for all nodes is identically distributed; therefore, we drop the subscripts corresponding to the particular node under investigation, and rather consider the performance of a node that *hits* (i.e., interferes with or is heard by) a particular number of other nodes when he transmits. We modify our notation to let p_k be the transmission probability of a node that hits k nodes when he transmits (including himself and his partner) and γ_k to represent the throughput for a node which hits k nodes. Making the assumption that both nodes of a partnership hit the same number of nodes (this is not valid, but our simulations show it to be a reasonable simplifying assumption), or at least that they use the same transmission probability, we obtain the following expression for the throughput:

$$\gamma_k = I p_k(1-p_k) \tag{3}$$

where we have collected the interference terms from nodes other than the node itself and his partner into the interference factor I. (In fact, $1-I$ is the interference encountered by a node, and perhaps we should call I the noninterference factor!) We can think of this factor as background interference. If we assume that the interference encountered at any node is independent of the number of nodes that he hits, then the expected throughput for any node in the network, γ_{node}, is given by

$$\gamma_{\text{node}} = I \sum_{k=2}^{n} h_k \gamma_k \tag{4}$$

where h_k is the probability that a node hits k nodes when he transmits (note that he always hits himself and his partner).

III. THE INTERFERENCE FACTOR

I is the product of terms corresponding to the interference generated by the nodes that are heard by the destination. We can group these terms depending on the number of nodes that the source of the interference hits when he transmits. We define I_k to be the total interference contribution of nodes that hit k nodes when they transmit. Let us call a node that hits k others when he transmits a "k-hitter." Then

$$I_k = \sum_{j=0}^{n-2} \Pr\{\text{a node hears } j\ k\text{-hitters}\} (1-p_k)^j. \tag{5}$$

The total interference will then be the product of these factors.

$$I = \prod_{k=3}^{n} I_k. \tag{6}$$

We define H_j^k to be the probability that an arbitrary node hears j k-hitters. We can evaluate this as follows:

$$H_j^k = \sum_{l=j}^{n-2} \Pr\{\text{Total of } l\ k\text{-hitters}\} \Pr\{\text{hear } j \text{ of } l\}$$

$$= \sum_{l=j}^{n-2} \binom{n-2}{l} (h_k)^l (1-h_k)^{n-2-l}$$

$$\cdot \binom{l}{j} \left(\frac{k-2}{n-2}\right)^j \left(1-\frac{k-2}{n-2}\right)^{l-j}. \tag{7}$$

We thus have

$$I_k = \sum_{j=0}^{n-2} \sum_{l=j}^{n-2} \binom{n-2}{l} (h_k)^l (1-h_k)^{n-2-l}$$

$$\cdot \binom{l}{j} \left(\frac{k-2}{n-2}\right)^j \left(1-\frac{k-2}{n-2}\right)^{l-j} (1-p_k)^j. \tag{8}$$

We can switch the order of summation to get

$$\begin{aligned} I_k &= \sum_{l=0}^{n-2} \binom{n-2}{l} (h_k)^l (1-h_k)^{n-2-l} \\ &\cdot \sum_{j=0}^{l} \binom{l}{j} \left(\frac{k-2}{n-2}\right)^j \left(1-\frac{k-2}{n-2}\right)^{l-j} (1-p_k)^j \\ &= \sum_{l=0}^{n-2} \binom{n-2}{l} (h_k)^l (1-h_k)^{n-2-l} \\ &\cdot \left[\frac{k-2}{n-2}(1-p_k) + \left(1-\frac{k-2}{n-2}\right)\right]^l \\ &= \sum_{l=0}^{n-2} \binom{n-2}{l} (h_k)^l \left(1-\frac{k-2}{n-2}p_k\right)^l (1-h_k)^{n-2-l} \\ &= \left[h_k\left(1-\frac{k-2}{n-2}p_k\right) + (1-h_k)\right]^{n-2} \\ &= \left[1-h_k\left(\frac{k-2}{n-2}\right)p_k\right]^{n-2}. \end{aligned} \tag{9}$$

Thus, the interference factor I is given by

$$I = \prod_{k=3}^{n} \left(1-\frac{k-2}{n-2}h_k p_k\right)^{n-2}. \tag{10}$$

For large n, we can use the exponential approximation to find

$$\begin{aligned} I &\approx \prod_{k=3}^{n} e^{-(k-2)h_k p_k} \\ &= e^{-\sum_{k=3}^{n}(k-2)h_k p_k}. \end{aligned} \tag{11}$$

The nodal throughput for an n-node network is given by

$$\gamma_{\text{node}} = \prod_{k=3}^{n} \left(1-\frac{k-2}{n-2}h_k p_k\right)^{n-2} \sum_{k=2}^{n} h_k p_k (1-p_k). \tag{12}$$

We now use this model to investigate the throughput properties of several configurations.

IV. COMPLETELY CONNECTED TOPOLOGIES

One approach to satisfying an arbitrary random traffic matrix is to give every node sufficient transmission power so that all the nodes in the network hear when any one transmits. This corresponds to the model of [1], [2], [12] and the total network throughput will therefore approach $1/e$. We proceed to show that our approach is consistent with this result.

Since the environment for each node is identical, we assume $p_i = p$. The number of nodes that can interfere with a given transmission is $n-1$, i.e., $h_n = 1$. The throughput for each node is

$$\begin{aligned} \gamma_{\text{node}} &= \left(1-\frac{n-2}{n-2}p\right)^{n-2} p(1-p) \\ &= p(1-p)^{n-1} \end{aligned} \tag{13}$$

which is the well-known slotted ALOHA result and reduces to $1/ne$ for $p = 1/n$ and large n.

V. ADJUSTABLE TRANSMITTER POWER

Another approach for arbitrary traffic matrices is to limit the power of each transmitter so that it exactly reaches its destination (again assuming that reception is a two-state process, either you can or cannot hear a transmission) [14]. In Fig. 1 we show a randomly generated two-dimensional network of ten nodes connected in this manner; the lines joining pairs of nodes represent the traffic matrix and, hence, the transmission radii (e.g., nodes 3 and 9 are a communicating pair).

A. Analytical Model

We now proceed to find the hitting distribution h_k. Consider an arbitrary node P in the network and rank the $n-1$ other nodes in order of their distance from P. If P is paired with a node in the $(k+1)$st position in this list (i.e., his kth neighbor), he will interfere with (hit) exactly $k+1$ nodes when he transmits. By assumption, P is equally likely to be paired with any of the nodes, and so the hitting distribution is given by

$$h_k = \frac{1}{n-1} \qquad k = 2, 3, \cdots, n. \tag{14}$$

We must now select an appropriate set of values for p_k. In order for the sum of the exponent of (11) to be bounded, p_k must be $O(1/k)$. We thus select $p_k = 1/k$ and note that this somehow corresponds to the optimality condition $G = 1$ in [2]. With these values for p_k and h_k, the sum in the exponents of (11) is

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=3}^{n} \frac{k-2}{k} \frac{1}{n-1} \\ &= \lim_{n\to\infty} \left[\sum_{k=3}^{n} \frac{1}{n-1} - \frac{2}{n-1}\sum_{k=3}^{n} \frac{1}{k}\right] \\ &= \lim_{n\to\infty} \left[\frac{n-2}{n-1} - \frac{O(\log n)}{n-1}\right] \\ &= 1 \end{aligned} \tag{15}$$

and so the interference factor will be

$$I = \frac{1}{e} \qquad \text{(for large networks)}. \tag{16}$$

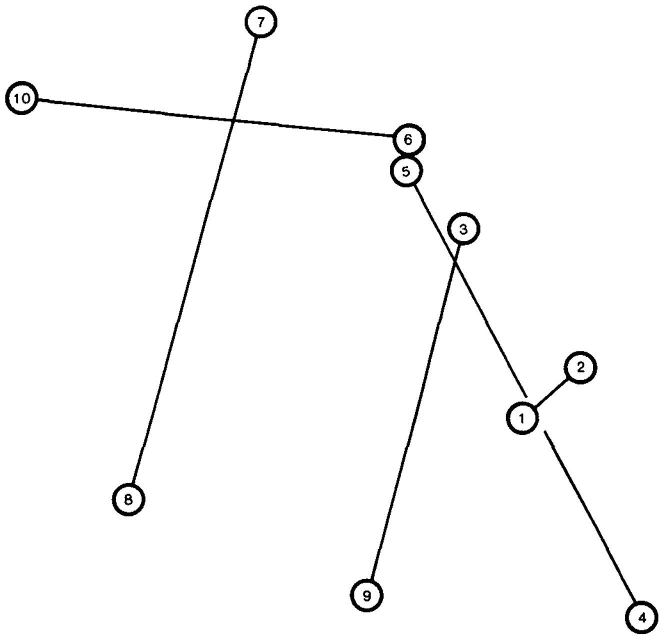

Fig. 1. Ten node limited power network.

We can now evaluate (12) to obatin the total network throughput.

$$\gamma_{\text{node}} = \frac{1}{n-1}\left[\prod_{k=3}^{n}\left(1-\frac{k-2}{n-2}\frac{1}{k}\frac{1}{n-1}\right)^{n-2}\right] \cdot \left[\sum_{k=2}^{n}\frac{1}{k}\left(1-\frac{1}{k}\right)\right]. \tag{17}$$

Since the throughput for each node is identically distributed, the total network throughput γ will be given by $n\gamma_{\text{node}}$.

$$\gamma = \frac{n}{n-1}\left[\prod_{k=3}^{n}\left(1-\frac{k-2}{n-2}\frac{1}{k}\frac{1}{n-1}\right)^{n-2}\right] \cdot \left[\sum_{k=2}^{n}\frac{1}{k}-\frac{1}{k^2}\right]. \tag{18}$$

Summing these series [4] and performing some algebraic manipulation, we find that the asymptotic behavior for large networks is given by

$$\lim_{n\to\infty} \gamma = \frac{\log(n) + C - \frac{\pi^2}{6}}{e} \tag{19}$$

where C is Euler's constant. This can be approximated by

$$\gamma \approx \frac{\log(n) - 1}{e}. \tag{20}$$

The above results were derived with no reference to the dimensionality of the network. We can therefore achieve a throughput *logarithmically* proportional to the network size for all networks satisfying a uniform traffic pattern by exact adjustment of transmission range. It must be pointed out, however, that the throughput for all pairs of nodes in the network is not the same. Nodes that are close together (and, thus, have high transmission probabilities since they do not interfere with many other nodes) will achieve higher throughputs than those that are far apart (recall that the background interference is assumed to be uniform for all nodes in the network). Even the node with the smallest throughput (in the worst case this node will hit $n-2$ other nodes) will have a throughput of $1/ne$ for large networks, which is the same as that for the fully connected case (in which *every* node achieves a throughput of $1/ne$). Thus, the node experiencing the worst performance will be doing no worse than for the fully connected case, whereas nodes close together will far exceed this throughput.

B. Simulation

In order to check the validity of this model, we developed a simulation program to compute the throughputs for these networks. This program operates as follows (described for a two-dimensional network).

A random network is generated with points uniformly distributed inside the unit circle. Pairs are then randomly assigned; in fact, we pair node 1 to node 2, node 3 to node 4, and so on (this being a perfectly random pairing). With this pairing, the transmission radii are determined so that communication can take place, and the adjacency matrix is computed. We then compute the transmission probability for a node to be the reciprocal of the number of nodes within range of that node. From this we can compute the success probabilities for each node and, hence, the network throughput. These data are then averaged over several runs (i.e., over several random networks and pairings).

In Fig. 2 we plot the model and simulation results for one-dimensional networks averaged over 50 networks for each data point. We see excellent agreement between model and simulation. The agreement is less good for small networks since edge effects are significant (i.e., a large proportion of the nodes are close to the edge of the network). Nodes near the edge of the network suffer less interference than is predicted by our assumption of the background interference being constant. As the number of nodes increases, these edge effects become proportionately less important. In the simulation, the transmission probability was $p = 1/k$, where k is the number of nodes hit by the transmission and partners not necessarily using the same k.

Fig. 3 shows similar results for the two-dimensional case. We again notice good agreement between the model and the simulation results for large n. In two dimensions the model requires larger networks before the agreement is good, due to the higher proportion of nodes on the edge of the area which suffer less interference. It is for this reason that the simulation results exceed that predicted by the model for small networks.

C. Other Transmission Policies

In addition to using the transmission policy of $p = 1/k$, we have also investigated several others, as outlined below. The performance of the following schemes is shown in Figs. 4-6.

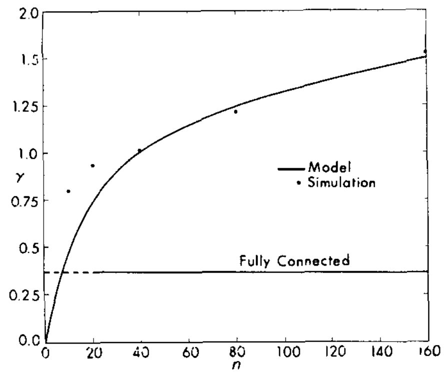

Fig. 2. One-dimensional random network–throughput.

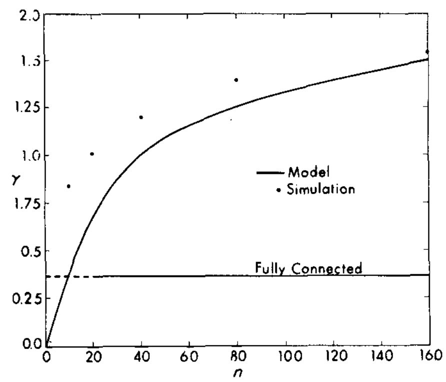

Fig. 3. Two-dimensional random network–throughput.

1) *Fixed Transmission Probability:* If we use as fixed transmission probability (independent of the hitting degree and the network size), the throughput as a function of network size rapidly falls to zero since too much interference is generated. Substituting fixed $p_k = p$ and $h_k = 1/n - 1$ into our model (12), we have

$$\gamma = np(1-p)e^{-\frac{p}{n-1}\sum_{k=3}^{n} k-2}$$

$$= np(1-p)e^{-p(n-2)/2}. \tag{21}$$

As an example, we set $p = 1/2$ and find

$$\gamma = \frac{n}{4} e^{-(n-2)/4}. \tag{22}$$

We plot this in Fig. 4, along with corresponding simulation results.

2) *Hearing Degree:* We also tried using the hearing degree rather than the hitting degree for determining the transmission probability. Fig. 5 shows the performance for the case where a node uses its own hearing degree to determine the transmission probability. We see that the throughput is independent of the network size and appears to be constant at $2/e$. A justification of this is that the number heard has mean $n/2$ and that the distribution has a sharp peak for large n (see the Appendix). Each node, therefore, hears $n/2$ nodes, each of which transmits with probability $2/n$. The network throughput is thus

$$\gamma = n\frac{2}{n}\left(1 - \frac{2}{n}\right)^{\frac{n}{2}} \approx \frac{2}{e} \approx 0.74. \tag{23}$$

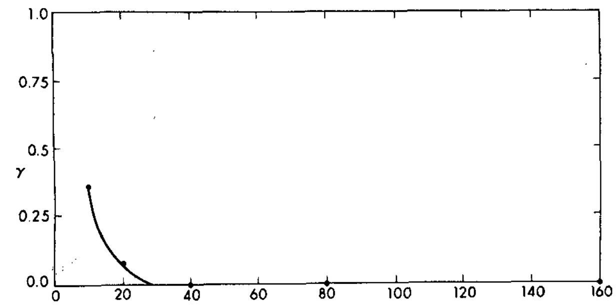

Fig. 4. Fixed transmission probability.

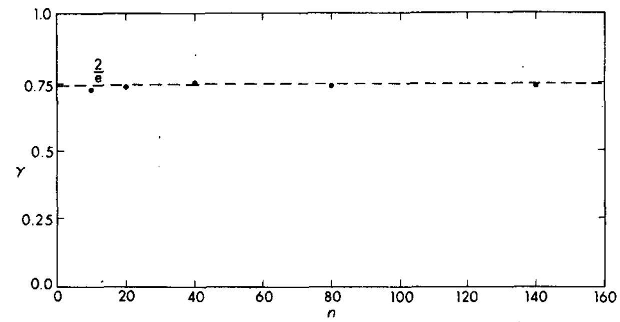

Fig. 5. Transmission probability based on hearing degree.

3) *Estimated Degree:* From a practical (implementation) point of view, it may be difficult for a node to determine exactly how many nodes hear when it transmits. We tried using an estimate of the hitting degree, equal to the number expected to be within range based on the transmission power and density of nodes (both quantities would probably be available to a node in a real network). Fig. 6 shows the performance of this scheme and we find, as expected, that the throughput grows logarithmically with the network size (note that for this case both nodes of a partnership will use the same transmission probability). We note that the performance is not quite as good for this scheme as when we used the actual hitting degree. This is probably mainly due to edge effects where the nodes actually have lower degrees than would be expected (and also suffer from less interference).

VI. BEST TRAFFIC MATRIX

In the previous discussion the transmission range was determined by the traffic matrix. Since we did not allow multihop paths, we required that the transmission power of a node be exactly sufficient to reach his communication partner. By changing the traffic matrix we can therefore further reduce the transmission ranges. We study this problem in this section, attempting to answer the following question: *for a random placement of nodes, what traffic matrix allows the highest traffic levels to be supported* [13]? We note here that we only need consider one-hop communication, since we could improve any multihop configuration (to achieve a higher throughput in terms of end-to-end messages) by considering each hop of the message to be a separate message. The performance of the best traffic matrix is somehow a measure of the network topology. It corresponds to the total capacity (summed over all links) in a traditional network, since the best traffic matrix for a traditional network is to send traffic to all neigh-

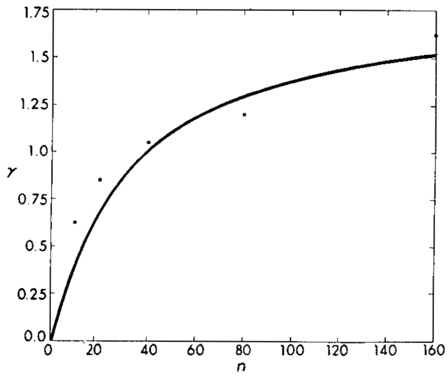

Fig. 6. Transmission probability based on an estimate of hitting degree.

bors at a rate equal to the capacity of the link. The performance that we obtain will therefore be an upper bound on the performance obtainable over the set of all traffic matrices. As we saw above, we can obtain a throughput proportional to the logarithm of the number of nodes for a *random* traffic matrix. It is commonly the case that the traffic requirements will exhibit some locality and we are therefore interested to see if we can obtain better than logarithmic performance for traffic matrices that exhibit locality. In particular, our bounds are for very local traffic. In a real network we expect that the traffic requirements will be between random and very local, and we thus expect that the performance will be between logarithmic and the bound obtained below.

First, we develop some simple upper and lower bounds on the maximum throughput that can be attained under any traffic matrix. It is clear that the performance of the "best" traffic matrix will lie between these bounds. The determination of the true "best" traffic matrix is hard.

A. Simple Bounds on Performance

In this section we give simple upper and lower gounds on the performance for the best possible traffic matrix (BTM).

1) *Upper Bound:* If there were *no* interference between pairs of nodes, we would be able to achieve a performance equal to that obtainable by $n/2$ independent pairs. One independent pair is able to support a throughput of 1/2 (which is achieved for a transmission probability of 1/2) [1]. Thus,

$$\gamma_{BTM} \leqslant \frac{n}{4}. \tag{24}$$

2) *Lower Bound:* As a lower bound we consider how many nodes (of the n total) can be paired up without any of them causing interference to any other pairs (clean pairs). The unpaired nodes are assumed to generate no traffic. Consider a pair of nodes in the network, P and Q. If these are to communicate without causing any interference, Q must be P's nearest neighbor and P must also be Q's nearest neighbor.

a) *One Dimension:* Here we have n nodes randomly located on the unit line. For simplicity we approximate this by a Poisson process of density $\lambda(=n)$. (Note that this approximation is only good for $n \gg 1$.) Fig. 7 shows two points P and Q in this random network. Suppose Q is P's nearest neighbor. The distribution $F_X(x)$ of X (the length of the line $\overline{PQ}$) can be found by noting that for Q to be P's nearest neighbor, there must be no point within a distance x on either side of P (see also [7], [11]). Thus, the distribution of neighbor distance is given by

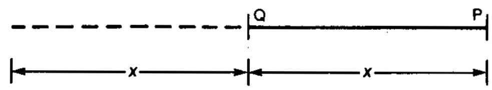

Fig. 7. No interference in one-dimensional network.

$$F_X(x) = \Pr\{X \leqslant x\} = 1 - \Pr\{X > x\} = 1 - e^{-2\lambda x}.$$

The density is thus

$$f(x)\,dx = 2\lambda e^{-2\lambda x}\,dx. \tag{25}$$

For no interference we require that there be no point closer to Q than P. Thus, if Q is to the left of P as shown in the figure, there must be no point within a distance x to the left of Q. The probability of finding no point there, f, is

$$f = e^{-\lambda x} \tag{26}$$

so the probability that a point is a member of a clean pair, g, is

$$g = \int_0^\infty 2\lambda e^{-2\lambda x} e^{-\lambda x}\,dx = \int_0^\infty 2\lambda e^{-3\lambda x} = \tfrac{2}{3}. \tag{27}$$

We see, then, that we can find a traffic matrix which can support (2/3) ($n/2$) clean pairs, which will allow a throughput of (2/3) ($n/4$) = $n/6$. Since this is readily achievable, it is clearly a lower bound on the performance of the "best" traffic matrix.

Letting $\gamma_{BTM}{}^k$ represent the throughput of the best possible configuration for a k-dimensional network, we have

$$\frac{n}{6} \leqslant \gamma_{BTM}{}^1 \leqslant \frac{n}{4}. \tag{28}$$

b) *Two Dimensions:* We now consider the two-dimensional analog. Considering Fig. 8, let Q be P's nearest neighbor. We assume that P and Q are randomly located in the unit circle by a Poisson process of parameter λ. (This model is not exact, as in fact we place *precisely* n points in a unit circle, but for large n it is a good approximation.) By analogy to the one-dimensional case, the density $f(x)$ of the length of the line $\overline{PQ}$ is

$$f(x)\,dx = 2\lambda\pi x e^{-\lambda \pi x^2}\,dx. \tag{29}$$

For no interference we require that there be no point closer to Q than P. That is, there is to be no point in the shaded area A

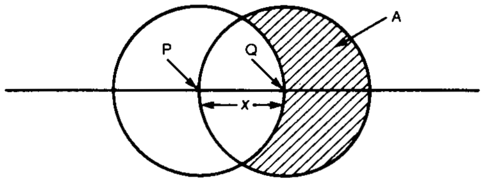

Fig. 8. No interference in two dimensions.

encircling Q (there is no point in the circle around P since Q is the nearest neighbor). This area can be found to be

$$A = x^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = 1.913x^2. \tag{30}$$

The probability of finding no point in this area is $e^{-\lambda A}$. So the probability that a point is a member of a clean pair, g, is

$$g = \int_0^\infty 2\lambda\pi x e^{-\lambda A} e^{-\lambda\pi x^2}\, dx = \frac{\pi}{\pi + 1.913} = 0.622. \tag{31}$$

This result can also be found in [3].

Thus we can find a traffic matrix allowing a throughput of $0.62n/4$, which is therefore a lower bound. Combining these equations we have the following relationship:

$$\frac{0.62n}{4} \leqslant \gamma_{\mathrm{BTM}}{}^2 \leqslant \frac{n}{4}. \tag{32}$$

We see, therefore, that we have found *linear* upper and lower bounds for the performance of the best possible traffic matrix, and that the lower bound is achievable. So, by appropriate selection of the traffic matrix and transmission range, we can achieve throughputs linearly proportional to the number of nodes in the network. Motivated by this greatly improved performance, we studied using small transmission ranges for multihop networks in [9], [15]. We found that for two-dimensional networks this linear performance, combined with the fact that we must travel a distance (number of hops) proportional to the square root of the number of nodes in the network, results in a throughput proportional to the square root of the number of nodes in the network.

B. Case Studies

In order to validate the above results, we now present some traffic patterns that achieve performance between these bounds. Since determination of the truly optimal traffic matrix is a hard problem, we look at some specific connection strategies allowing us to achieve high throughputs. For some of these cases we can proceed with the analysis outlined earlier, but in all cases we give simulation results.

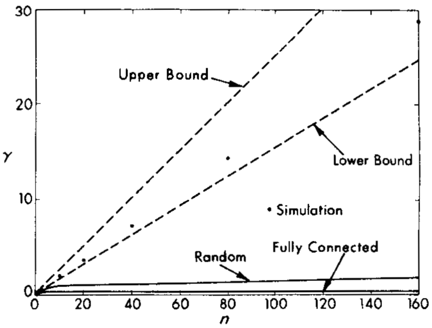

Fig. 9. Two-dimensional NUN–comparison to bounds.

C. Nearest Unpaired Neighbor (NUN) in Two Dimensions

In this section we consider random two-dimensional networks and give a low interference connection strategy, the behavior of which exceeds the lower bound on optimal throughput given in Section VI-A-2-b.

Algorithm A, below, describes the policy for pairing nodes in the nearest unpaired neighbor scheme.

Algorithm A–Nearest Unpaired Neighbor (NUN):

1) Generate the random network, consisting of an even number of nodes.

2) Mark all nodes as unpaired.

3) Find the two closest unpaired nodes and connect them; mark them as paired.

4) If all nodes are paired, we have finished; otherwise, return to step 3.

The traffic pattern generated by this algorithm is satisfied by giving each node sufficient power to exactly reach his destination. Fig. 9 shows the performance of this scheme in comparison to the bounds developed earlier.

D. Nearest Unpaired Neighbor (NUN) in One Dimension

This is the one-dimensional equivalent of the two-dimensional scheme outlined above. Fig. 10 shows the simulation results for networks using transmission probabilities based on the node's hitting degree.

In the following section we consider a simpler version of this, in which every node is connected (adjoined) to his left (or right) neighbor.

E. Adjoining in One Dimension (ADJ)

For this scheme we randomly locate n points on the unit line and then connect adjacent pairs starting from one end. In Fig. 12 we show the performance for this scheme. We notice that the performance is very similar to the NUN scheme outlined above. Let us analyze this case.

1) *The Hitting Distribution:* We know the distribution of the distance to the neighbor on your left (or right) and we must determine how many points are expected to fall in this distance on the other side of the connection, this being the number of points that will hear you. The distribution of the neighbor distance, x, is

$$f(x)\, dx = \lambda e^{-\lambda x}\, dx. \tag{33}$$

The points that you hit are precisely those that fall in a dis-

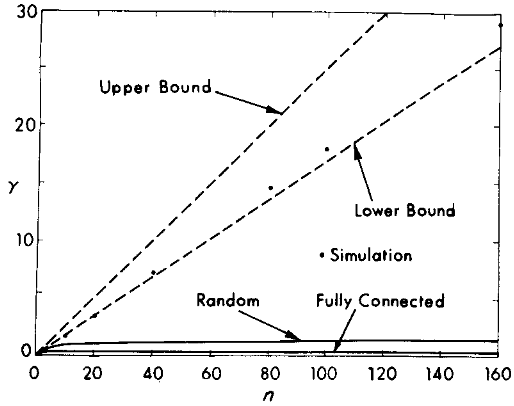

Fig. 10. One-dimensional NUN–comparison to bounds.

tance x on your right (left). The number of points falling in this distance on the other side is Poisson distributed; thus

$$\Pr\{i \text{ in a distance } x\} = e^{-\lambda x}\frac{(\lambda x)^i}{i!} \tag{34}$$

so we have

$$h_{i+2} = \int_0^\infty \frac{(\lambda x)^i}{i!}e^{-\lambda x}\lambda e^{-\lambda x}\,dx = \int_0^\infty \frac{\lambda^{i+1}x^i}{i!}e^{-2\lambda x}\,dx$$
$$= (1/2)^{i+1}. \tag{35}$$

We can derive this distribution in an alternate manner without having to rely on the exponential or Poisson distributions as follows. In Fig. 11, suppose that node P (whose partner is Q) hits i *excess* (i.e., other than himself and his partner) nodes. Let R be the first point on the left that cannot hear P. For this to happen, P must be to the right of the midpoint of $\overline{QR}$; the probability of this event is 1/2. If x is the distance from P to Q, then consider a point P' at a distance x to the left of P. Now all the i excess points must fall to the left of the midpoint of $\overline{QP'}$ (i.e., to the left of P). The probability of this event is $(1/2)^i$. Thus, the probability that P hits i points is

$$h_{i+2} = (1/2)^{i+1}. \tag{36}$$

2) *Evaluation of Interference:* We can determine this interference factor directly from (11).

$$I = e^{-\sum_{k=3}^{n}(k-2)h_k p_k}. \tag{37}$$

As before, we use $p_k = 1/k$, and using the hitting distribution found above, i.e., $h_k = (1/2)^{k-1}$, we find

$$I = \exp - \left\{\sum_{k=3}^{n}(k-2)(1/2)^{k-1}\frac{1}{k}\right\}$$
$$= \exp - \left\{\sum_{k=3}^{n}(1/2)^{k-1} - 2\sum_{k=3}^{n}(1/2)^{k-1}\frac{1}{k}\right\}.$$

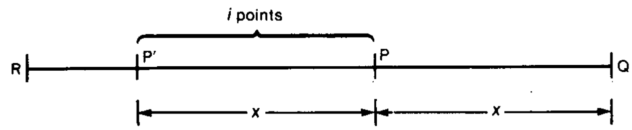

Fig. 11. One-dimensional ADJ–hitting degree.

These sums can be evaluated to give

$$I \approx \exp - \left\{\tfrac{1}{2} + 4\left(\log\left(1 - \tfrac{1}{2}\right) + \tfrac{1}{2} + \tfrac{1}{8}\right)\right\} \quad \text{(for large } n\text{)}. \tag{38}$$

This expression gives

$$I = e^{-3}e^{-4\log(1/2)} = 2^4e^{-3} = 0.797. \tag{39}$$

From (3) and (4), the throughput for node j, γ_j, is given by

$$\gamma_j = I\sum_{i=2}^{n} h_i \frac{1}{i}\left(1 - \frac{1}{i}\right)$$
$$= 2I\left[\sum_{i=2}^{n}\frac{1}{i}(1/2)^i - \sum_{i=2}^{n}\frac{1}{i^2}(1/2)^i\right]$$
$$\approx 2I\left[\log(2) + \frac{[\log(2)]^2}{2} - \frac{\pi^2}{12}\right] \quad \text{(for large } n\text{)}$$
$$= 0.1756. \tag{40}$$

Thus, the total network throughput γ is given by

$$\gamma \approx 0.176n \qquad \text{(for large } n\text{)}. \tag{41}$$

Fig. 12 shows the throughput predicted by this model and simulation results from the "hitting degree" transmission scheme. We see very good agreement between analytical and simulation results. We do not analyze the two-dimensional network but, due to the similarity of the curves found by simulation, anticipate that a similar result applies.

VII. CONCLUSIONS

In this paper we presented a general model for determining the throughput of a homogeneous single-hop random slotted ALOHA network. We then proceeded to consider various examples. We first used our model to give a new derivation of the known capacity of $1/e$ for fully connected networks. We then introduced the idea of restricting transmission range and found that we could achieve a throughput proportional to the *logarithm* of the number of nodes in the network for a uniform traffic matrix. We then considered the problem of finding the "best" traffic matrix. We first found some simple bounds and then used our model to predict the performance for a traffic matrix that seemed to have low interference and found that a throughput proportional to the number of nodes in the network could be achieved. Since traffic requirements in real networks tend to exhibit some locality, the performance would be between the logarithmic and linear performance that we have demonstrated for random and very local traffic.

Finding that range reduction results in such a dramatic in-

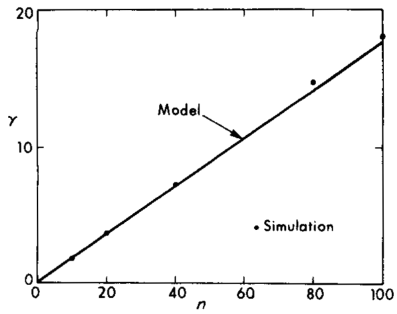

Fig. 12. One-dimensional ADJ.

crease in performance leads us to suspect that benefits might also be obtained by reducing transmission range in multihop networks. We do not investigate multihop networks in this paper, but as reported in [9], [15], reducing range in multihop networks is also beneficial.

We believe that the model developed in this paper can be applied to other homogeneous single-hop networks. In addition, it can be used to evaluate the one-hop throughput (i.e., ignoring the fact that messages may be transmitted more than once in their path from source to destination) of an arbitrary *multihop* network.

APPENDIX

DERIVATION OF HEARING DISTRIBUTION

We wish to evaluate the probability that an arbitrary node in the network will hear another node. Let us call the probability of hearing a particular k-hitter α_k. Assuming that the "hits" of this node are uniformly distributed over the set of nodes, we can evaluate α_k.

$$\alpha_k = \frac{\binom{n-3}{k-3}}{\binom{n-2}{k-2}} = \frac{k-2}{n-2}. \tag{A1}$$

By unconditioning on k, we can evaluate the probability α that we hear any particular other node.

$$\begin{aligned} \alpha &= \sum_{k=2}^{n} h_k \frac{k-2}{n-2} \\ &= \sum_{k=2}^{n} \frac{1}{n-1} \frac{k-2}{n-2} \\ &= \frac{(n-2)(n-1)}{2} \frac{1}{(n-1)(n-2)} = \frac{1}{2}. \end{aligned} \tag{A2}$$

Thus, the probability of hearing j other nodes, H_j (the hearing distribution), is the binomial distribution given below. (Note the subscript for H is $j + 2$ since a node always hears two others—himself and his partner.)

$$H_{j+2} = \binom{n-2}{j} \left(\tfrac{1}{2}\right)^j \left(\tfrac{1}{2}\right)^{n-2-j} \quad j = 0, 1, \cdots, n-2. \tag{A3}$$

Thus, the mean number of nodes is $(n/2) + 1$, including the node himself and his partner.

REFERENCES

[1] N. Abramson, "The ALOHA system—Another alternative for computer communications," in *AFIPS Conf. Proc., Fall Joint Comput. Conf.*, 1970, vol. 37, pp. 281–285.

[2] ——, "The throughput of packet broadcasting channels," *IEEE Trans. Commun.*, vol. COM-25, pp. 117–128, Jan. 1977.

[3] H. Dewitt, "The theory of random graphs with applications to the probabilistic analysis of optimization algorithms," Ph.D. dissertation, Dep. Comput. Sci., Univ. California, Los Angeles, 1977.

[4] L. B. W. Jolley, *Summation of Series*. New York: Dover, 1961.

[5] R. E. Kahn, "The organization of computer resources into a packet radio network," *IEEE Trans. Commun.*, vol. COM-25, pp. 169–178, Jan. 1977.

[6] R. E. Kahn, S Gronemeyer, J. Burchfiel, and R. Kunzelman, "Advances in packet radio technology," *Proc. IEEE*, vol. 66, pp. 1468–1496, Nov. 1978.

[7] M. G. Kendall and P. Moran, *Geometrical Probability*. London, England: Griffin, 1963.

[8] F. F. Kuo and N. Abramson, "Some advances in radio communications for computers," in *Dig. Papers, 7th Annu. IEEE Comput. Soc. Int. Conf.*, San Francisco, CA, Feb. 1973.

[9] L. Kleinrock and J. A. Silvester, "Optimal transmission radii in packet radio networks or why six is a magic number," in *Proc. Nat. Telecommun. Conf.*, Birmingham, AL, Dec. 1978.

[10] S. Lam, "Packet switching in a multi-access broadcast channel with applications to satellite communication in a computer network," Comput. Syst. Modeling Anal. Group, School Eng. Appl. Sci., Univ. California, Los Angeles, UCLA-ENG-7429, Apr. 1974 (also, Ph.D. dissertation, Dep. Comput. Sci., U.C.L.A.).

[11] S. A. Roach, *The Theory of Random Clumping*. London, England: Methuen, 1968.

[12] L. G. Roberts, "ALOHA packets system with and without slots and capture," *Comput. Commun. Rev.*, vol. 5, pp. 28–42, Apr. 1975.

[13] J. A. Silvester and L. Kleinrock, "Capacity of neighbor communications in packet radio networks," presented at the 12th Hawaii Int. Conf. Syst. Sci., Jan. 1979.

[14] ——, "On the capacity of one-hop ALOHA packet radio networks with adjustable transmitter power," presented at the Pacific Telecommun. Conf., Jan. 1980.

[15] ——, "On the capacity of multihop slotted ALOHA networks with regular structure," this issue, pp. 974–982.

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John A. Silvester (M'79), for a photograph and biography, see this issue, p. 982.

★

Leonard Kleinrock (S'55–M'64–SM'71–F'73), for a photograph and biography, see this issue, p. 982.