Output Processes in Contention Packet Broadcasting Systems

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Abstract—**The processes consisting of the packet interdeparture times in contention-type packet broadcasting systems are studied under the heavy-traffic assumption. The channel access protocols considered include slotted and unslotted ALOHA and carrier-sense-multiple-access (CSMA) with and without collision detection. Through analysis of the channel activity cycle, the distribution, mean, and coefficient of variation of the packet interdeparture times are explicitly derived. Taking the reciprocal of the mean interdeparture time, we obtain the channel throughput. Cases with dissimilar users are mainly considered, and systems of statistically identical users are treated as special cases.**

I. Introduction

THIS paper presents an analysis of packet interdeparture times (i.e., intervals between two consecutive successful transmissions) for a number of channel access protocols encountered in packet broadcasting communication systems such as packet radio networks and local-area computer networks. Specifically, we are interested in the average $\bar{X}$ and the coefficient of variation $C^2 = \text{Var}\ [X]/\bar{X}^2$ of the *packet interdeparture time* X for given protocols. Throughout the paper we assume the constant packet transmission time to be 1 as the unit of time. Then, $S = 1/\bar{X}$ is equivalent to the *channel throughput,* which can alternatively be obtained as the ratio of the average time that the channel is used for successful transmission in a cycle of channel usage to the average cycle duration. Our approach is to calculate $\bar{X}$ and C^2 from the distribution of X. We also make the *heavy-traffic assumption* that all users contain packets all the time.

The channel access protocols we consider here include:

a) Pure ALOHA [1]
b) Slotted ALOHA [7]
c) Slotted carrier-sense-multiple-access (CSMA) [4], [10]
d) Slotted CSMA with collision detection (CSMA/CD) [11]
e) Unslotted CSMA [4], [10]
f) Unslotted CSMA with collision detection.

(Each protocol model is described individually below.) In all models, we assume the *memoryless property* that whenever a user experiences an idle (nontransmitting) period, he renews his action independently of the past happenings. Then we can find the distribution of X explicitly for all of the above-listed protocols. We may assume that each user has a different value for its transmission parameter, such as the probability of transmission in a slot. Such a case occurs, for example, in the priority-based ordering of users, or the adaptive self-adjustment of parameter values according to the imposed load. Below we mainly consider these cases of nonidentical users, and treat systems of identical users as special cases.

Paper approved by the Editor for Computer Communication of the IEEE Communications Society. Manuscript received September 3, 1982; revised April 17, 1985. This work was supported by the Defense Advanced Research Projects Agency under Contract MDA 903-82-C-0064.

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Using the mean and variance of the packet interdeparture times from the system, we can get the average and variance of the number of successful transmissions in a given long interval. Furthermore, we can relate these system-wide quantities to the means and covariances of the numbers of successful transmissions from the individual users. They can then be used to determine the coefficients in the diffusion process approximation to the user's queue length distribution [9].

As related work, we note Tobagi's analysis of packet interdeparture time based on the "linear feedback model" of slotted ALOHA and slotted CSMA [12].[1] For pure ALOHA, Ferguson [2] gives an approximation to the packet interdeparture time for a randomly selected user.

II. The Number of Successful Transmissions

Let $\{X^{(n)};\ n = 1, 2, \cdots\}$ be a sequence of packet interdeparture times (from the entire system) beginning at the end of an arbitrarily chosen successful transmission (let this instant be the time origin $t = 0$). For all memoryless protocols defined above, they are independent and identically distributed; their generic representation is X. Then

$$S^{(n)} = X^{(1)} + X^{(2)} + \cdots + X^{(n)} \qquad n = 1, 2, \cdots \tag{1}$$

defines the time at which the nth successful transmission completes. By definition, $\{S^{(n)};\ n = 1, 2, \cdots\}$ is a *renewal process*; see, for example, [3]. For time $t > 0$, let $D(t)$ be the number of successful transmissions completed during an interval $[0, t]$:

$$D(t) = \max\ \{n;\ S^{(n)} \leqslant t\}. \tag{2}$$

Now renewal theory tells us

$$\lim_{t\to\infty} \frac{\overline{D(t)}}{t} = \frac{1}{\bar{X}};\quad \lim_{t\to\infty} \frac{\text{Var}\ [D(t)]}{t} = \frac{\text{Var}\ [X]}{\bar{X}^3}. \tag{3}$$

Thus, the asymptotic behavior of $\overline{D(t)}$ and $\text{Var}[D(t)]$ can be obtained from $\bar{X}$ and $\text{Var}[X]$.

Next, let us assume that $M < \infty$ is the number of users in the system whose transmission parameters are not necessarily identical. They are indexed as $1, 2, \cdots, M$. Let $D_i(t)$ be the number of successful transmissions completed by user i during $[0, t]$ ($i = 1, 2, \cdots, M$). Let q_i be the probability that a successful transmission is achieved by user i; $\Sigma_{i=1}^{M} q_i = 1$. (In the case where all users have the same parameter value, we have $q_i = 1/M$.) Then it can be readily shown (see Appendix A) that the means and covariances of $[D_1(t), D_2(t), \cdots,$

[1] A referee of this paper has pointed out that, once the heavy traffic assumption is made, the whole analysis in [12] reduces immediately to a single equation similar to that given in the present paper. However, [12] only deals with slotted ALOHA and slotted CSMA systems of statistically identical users (without the heavy traffic assumption), while we consider other protocols and cases of nonidentical users as well (with the heavy traffic assumption).

$D_M(t)$] are given by

$$\overline{D_i(t)} = q_i \overline{D(t)}$$

$$\text{Cov}\ [D_i(t),\ D_j(t)] = q_i q_j \{\text{Var}\ [D(t)] - \overline{D(t)}\} + \delta_{ij} q_i \overline{D(t)} \qquad i,\ j = 1,\ 2,\ \cdots,\ M \quad (4)$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

We note that the dependence of $D_i(t)$ and $D_j(t)$ ($i \neq j$) comes from the fact that when user i is successful, user j is not, and vice versa. Therefore, by use of (3) and (4), we can obtain the asymptotic behavior of the numbers of successful transmissions by individual users.

We remark that if $\{X_i^{(n)};\ n = 1, 2, \cdots\}$ is a sequence of packet interdeparture times from user i ($i = 1, 2, \cdots, M$) beginning at $t = 0$ (which can be the completion time of some other user), then $\{S_i^{(n)};\ n = 1, 2, \cdots\}$, where $S_i^{(n)} = X_i^{(1)} + X_i^{(2)} + \cdots + X_i^{(n)}$ ($n = 1, 2, \cdots$), is a *delayed renewal process*, i.e., the distribution of $X_i^{(1)}$ is not identical to those of $X_i^{(2)}, X_i^{(3)}, \cdots$ which are identical and generically denoted by X_i. However, for $D_i(t) = \max\ \{n;\ S_i^{(n)} \leqslant t\}$, we still have the asymptotes

$$\lim_{t\to\infty} \frac{\overline{D_i(t)}}{t} = \frac{1}{\bar{X}_i};$$

$$\lim_{t\to\infty} \frac{\text{Var}\ [D_i(t)]}{t} = \frac{\text{Var}\ [X_i]}{\bar{X}_i^3} \qquad i = 1,\ 2,\ \cdots,\ M. \quad (5)$$

Substituting (3) and (5) into (4) for $i = j$, we get the relationship

$$S_i = q_i S; \quad 1 - C_i^2 = q_i(1 - C^2) \qquad i = 1,\ 2,\ \cdots,\ M \quad (6)$$

where $C_i^2 = \text{Var}\ [X_i]/\bar{X}_i^2$, and $S_i = 1/\bar{X}_i$ is the throughput of user i. In the case of identical users ($q_i = 1/M$), (6) reduces to

$$S = MS_i; \quad 1 - C^2 = M(1 - C_i^2) \qquad i = 1,\ 2,\ \cdots,\ M. \quad (7)$$

We note that (6) and (7) can also be derived by considering random splitting of a non-Poisson stream as shown in [5].

III. CASES OF IDENTICAL UNSUCCESSFUL TRANSMISSION PERIODS

In this section, we consider the packet interdeparture time X for a subset of memoryless protocols in a unified manner. To do so, we define the *transmission period* in a channel as the state where at least one user is transmitting or any transmission is being sensed. Also, the channel *idle period* is defined as the state where no users are transmitting or no transmissions are being sensed. Thus, the channel state alternates between the transmission and idle periods. (There can be two consecutive transmission periods with an idle period of duration 0 between them.) Also, each successful transmission period is preceded by a number of alternating idle periods and unsuccessful transmission periods.

Cases treated in this section are categorized such that successive unsuccessful transmission periods are identically distributed. In this category fall slotted systems (ALOHA, CSMA, CSMA/CD) of nonidentical users and unslotted systems of identical users. Below, we first discuss the distribution of X generally. Then we show two examples to which our analysis is applied.

Now, let K be the number of transmission periods included in X of which the last one is the only successful transmission. Let $I^{(k)}$ and $F^{(k)}$ be the durations of the kth idle period and kth unsuccessful transmission period, respectively, and T be the duration of the successful transmission period. Then, we have

$$X = \sum_{k=1}^{K-1} [I^{(k)} + F^{(k)}] + I^{(K)} + T. \quad (8)$$

Since we have assumed the memoryless property in protocol, the beginning of each idle period is a system renewal point (i.e., the behavior of the system after that point does not depend on what happened before that point). Therefore, $\{I^{(k)};\ k = 1, 2, \cdots\}$ are independent and identically distributed random variables whether users are identical or not; let I be a generic representation of the $I^{(k)}$'s. For the same reason, the sequence of $\{F^{(k)};\ k = 1, 2, \cdots\}$, consisting of $F^{(k)}$ following $I^{(k)}$, are independent variables. Furthermore, we assume that $\{F^{(k)};\ k = 1, 2, \cdots\}$ are identically distributed. So, let F be the generic representation of the $F^{(k)}$'s. Thus, a sequence of renewal cycle durations $\{I^{(k)} + F^{(k)};\ k = 1, 2, \cdots\}$ are independent and identically distributed as $I + F$. Also, $I^{(K)} + T$ is independent of the previous cycles and is distributed as $I + T$. By these arguments, we can compute the mean and variance of X directly as

$$\bar{X} = (\bar{K} - 1)(\bar{I} + \bar{F}) + \bar{I} + \bar{T}$$

$$\begin{aligned} \text{Var}\ [X] &= (\bar{K} - 1)\ \text{Var}\ [I + F] + (\bar{I} + \bar{F})^2\ \text{Var}\ [K] + \text{Var}\ [I + T] \\ &= \bar{K}\ \text{Var}\ [I] + (\bar{K} - 1)\ \text{Var}\ [F] + \text{Var}\ [T] + (\bar{I} + \bar{F})^2\ \text{Var}\ [K] \end{aligned} \quad (9)$$

where we have assumed that $F^{(k)}$ and T are independent of $I^{(k)}$.

If we denote by $I^*(s)$, $F^*(s)$, and $T^*(s)$ the Laplace transforms of the pdf's for I, F, and T, respectively, and denote by $K^*(z)$ the z-transform of the distribution of K, then under the same assumptions, the Laplace transform of the pdf for X, $X^*(s)$, is given by

$$X^*(s) = \sum_{k=1}^{\infty} [I^*(s)F^*(s)]^{k-1}[I^*(s)T^*(s)] \cdot \text{Prob}\ [K = k]$$

or

$$X^*(s) = \frac{T^*(s)}{F^*(s)} K^*[I^*(s)F^*(s)]. \quad (10)$$

Now, let γ be the probability of a successful transmission once it has been started by breaking the channel idle period (γ is a protocol-dependent function of M and other system parameters). Then, clearly, K has a geometric distribution

$$\text{Prob}\ [K = k] = (1 - \gamma)^{k-1}\gamma \qquad k = 1,\ 2,\ \cdots$$

$$K^*(z) = \frac{\gamma z}{1 - z(1 - \gamma)},$$

$$\bar{K} = \frac{1}{\gamma}; \quad \text{Var}\ [K] = \frac{1 - \gamma}{\gamma^2}. \quad (11)$$

From (10) and (11), we obtain the fundamental relationship

$$X^*(s) = \frac{\gamma T^*(s)}{\dfrac{1}{I^*(s)} - (1 - \gamma)F^*(s)} \quad (12)$$

Therefore, given a protocol, $\bar{X}$ and C^2 can be computed by (9) and (11) if we obtain γ and the means and variances of I, F, and T depending on the protocol. Also, by (12), $X^*(s)$ can be obtained from γ and the distributions of I, F, and T.

A. Pure ALOHA (Infinite Population)

For pure (or unslotted) ALOHA where all packets have the same length 1, we can analytically obtain the distribution of X (and C^2) only for an infinite population of users who collectively form a Poisson source of packet transmissions.

Let us consider an infinite population of users from which packets are transmitted such that the interarrival times are independent and exponentially (identically) distributed with mean $1/G$; see Fig. 1(a). Then, from its memoryless property, the channel idle time is exponentially distributed in the same way:

$$\text{Prob } [I \leqslant y] = 1 - e^{-Gy} \qquad y \geqslant 0; \quad I^*(s) = \frac{G}{G+s}$$

$$\bar{I} = \frac{1}{G}; \quad \text{Var } [I] = \frac{1}{G^2}. \tag{13}$$

The probability of a successful transmission of an idle-period-breaking packet is given by

$$\gamma = e^{-G}, \tag{14}$$

i.e., the probability of no other transmissions during its entire transmission period $T = 1$ ($T^*(s) = e^{-s}$). The duration of an unsuccessful transmission period F is analyzed in Appendix B, where we derive

$$F^*(s) = \frac{Ge^{-(s+G)}[1 - e^{-(s+G)}]}{(1 - e^{-G})[s + Ge^{-(s+G)}]},$$

$$\bar{F} = \frac{e^G - 1 - Ge^{-G}}{G(1 - e^{-G})}; \quad \text{Var } [F] = \frac{e^{2G}}{G^2} - \frac{2e^G}{G} - \frac{e^{-G}}{(1 - e^{-G})^2}. \tag{15}$$

Substituting (13)–(15) into (9), (11), and (12), we get

$$X^*(s) = \frac{Ge^{-(s+G)}[s + Ge^{-(s+G)}]}{s^2 + sG[1 + e^{-(s+G)}] + G^2e^{-2(s+G)}}$$

$$S = Ge^{-2G}; \quad C^2 = 1 + 2e^{-G} - 2e^{-2G} - 4Ge^{-2G}. \tag{16}$$

We note that the result for C^2 is new while the expression for S is given in [1]. For $G = 1/2$, which maximizes S, we have $S = 1/(2e) = 0.1839$ and $C^2 = 0.7415$.

B. Slotted CSMA and CSMA with Collision Detection

We now proceed to analyzed slotted CSMA where the slot size is equal to a, the ratio of the signal propagation delay to the packet transmission time. We consider CSMA/CD such that an unsuccessful transmission period lasts $b + a$, where $a \leqslant b \leqslant 1$; X is illustrated in Fig. 1(b). Thus, the case $b = 1$ corresponds to CSMA without collision detection.

Let user i start to transmit (after sensing any idle slot) with probability p_i independently of all others. Such a transmission is successful if none of the other users have started transmission at the same time. (After the first slot of transmission period, no other users start transmission because they sense the channel busy.) Thus, we have the expression

$$\gamma = U/(1 - E) \tag{17}$$

where

$$E = \prod_{i=1}^{M} (1 - p_i); \quad U = \sum_{i=1}^{M} p_i \prod_{\substack{j=1 \\ (j \neq i)}}^{M} (1 - p_j). \tag{18}$$

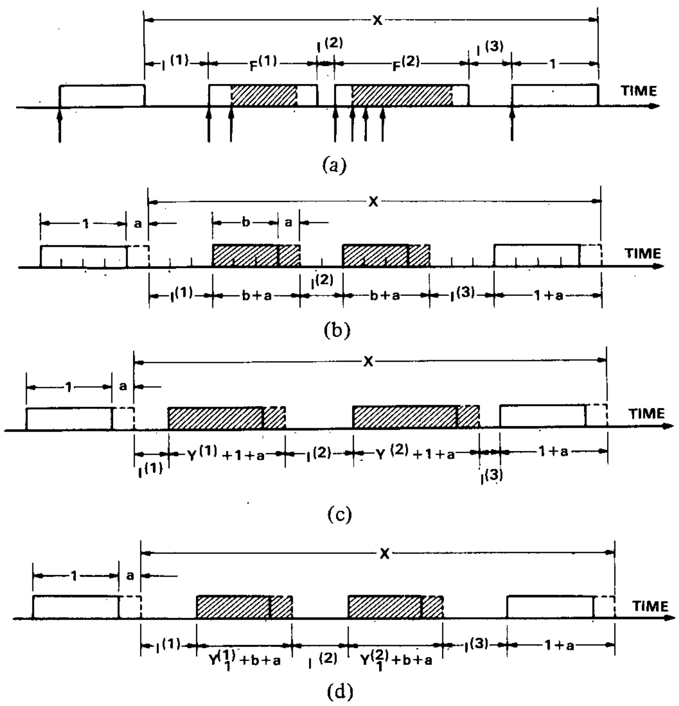

Fig. 1. Packet interdeparture times X. (a) Pure ALOHA. (b) Slotted CSMA with collision detection. (c) Unslotted CSMA. (d) Unslotted CSMA with collision detection.

The channel idle period is geometrically distributed as

$$\text{Prob } [I = na] = E^n(1 - E) \qquad n = 0, 1, 2, \cdots$$

$$I^*(s) = \frac{1 - E}{1 - e^{-as}E}. \tag{19}$$

The transmission periods are of constant length:

$$T = 1 + a; \quad F = b + a$$

or

$$T^*(s) = e^{-s(1+a)}; \quad F^*(s) = e^{-s(b+a)}. \tag{20}$$

Substituting (17), (19), and (20) into (9) and (11), we get

$$S = \frac{U}{a + U + b(1 - U - E)}$$

$$\text{Var } [X] = \frac{[a + b(1 - E)]^2}{U^2} + \frac{b^2E - (b+a)^2}{U}. \tag{21}$$

From (12), we also get

$$X^*(s) = \frac{Ue^{-s(1+a)}}{1 - e^{-as}E - e^{-s(b+a)}(1 - U - E)} \tag{22}$$

from which we have

$$\text{Prob } [X = 1 + a + na + k(b + a)]$$

$$= U\binom{n+k}{k}E^n(1 - U - E)^k; \quad n, k = 0, 1, 2, \cdots. \tag{23}$$

The implication of (23) should be clear, since n and k are the numbers of idle slots and unsuccessful transmission periods, respectively, experienced until the time of a successful transmission.

For individual users, the throughput S_i and the coefficient of variation of interdeparture times C_i^2 can be calculated by (6) with

$$q_i = p_i \prod_{\substack{j=1 \\ (j \neq i)}}^{M} (1-p_j)/U. \tag{24}$$

It can be shown that the maximum allowable throughput contour in the $\{p_i\}$ space is given by

$$a + b(1-E) = (a+b) \sum_{i=1}^{M} p_i. \tag{25}$$

In the case of identical users ($p_i = p$ for all i), (25) reduces to

$$(a+b)(1-pM) = b(1-p)^M \tag{26}$$

which was derived by Molle [6] in his study of the "local optimality" condition. It can be proved that with the value of p determined by (26), we have $C^2 < 1$. In the case of CSMA without collision detection ($b = 1$), we have

$$S = \frac{U}{1+a-E}; \quad C^2 = 1 - U\left[\frac{(1+a)^2 - E}{(1+a-E)^2}\right]. \tag{27}$$

The channel throughput S in (21) in the limit $M \to \infty$ with G fixed such that $aG = pM$:

$$S = \frac{aGe^{-aG}}{a + aGe^{-aG} + b(1 - aGe^{-aG} - e^{-aG})} \tag{28}$$

has been obtained in [11]. The result for the case of no collision detection ($b = 1$) was given incorrectly in [10] and was corrected in [4]. (The plot in [4], however, was for the result in [10]; the corrected plot was given by Molle [6]).

IV. Unslotted CSMA

We next consider unslotted CSMA where the propagation delay is again denoted by a; for an illustration of X, see Fig. 1(c). We assume that user i schedules his next transmission at an exponentially distributed time after he has sensed the channel idle. Let $1/g_i$ be the mean of this exponential distribution. Since the channel idle time I is the minimum of all user's scheduling delays, its distribution is given by

$$\text{Prob } [I \leq y] = 1 - \exp\left[-y \sum_{i=1}^{M} g_i\right] \qquad y \geq 0$$

$$\bar{I} = \frac{1}{\sum_{i=1}^{M} g_i}; \quad \text{Var } [I] = \frac{1}{\left(\sum_{i=1}^{M} g_i\right)^2}. \tag{29}$$

The terms in a sequence of $\{I^{(k)} + F^{(k)}; k = 1, 2, \cdots\}$ in (8) are, however, no longer identically distributed although they are independent. Also, the probability of success for an already started transmission differs from cycle to cycle, depending on which user initiates the transmission period. Let us look at these points more closely.

First, notice that the probability that user i among others begins transmission, breaking the channel idle period, is given by

$$\nu_i = \frac{g_i}{\sum_{j=1}^{M} g_j}; \quad \sum_{i=1}^{M} \nu_i = 1. \tag{30}$$

The probability of success in a cycle where user i initiates transmission period is then given by

$$\gamma_i = \exp\left[-a \sum_{\substack{j=1 \\ (j \neq i)}}^{M} g_j\right] \qquad i = 1, 2, \cdots, M. \tag{31}$$

For later use, let us denote by

$$\gamma = \sum_{i=1}^{M} \nu_i \gamma_i = \frac{\sum_{i=1}^{M} g_i \exp\left[-a \sum_{\substack{j=1 \\ (j\neq i)}}^{M} g_j\right]}{\sum_{i=1}^{M} g_i} \tag{32}$$

the probability that a transmission period is successful.

In an unsuccessful transmission period initiated by user i, let $Y(i)$ be the transmission start time of the last colliding packet. Its distribution is given by

$$\text{Prob } [Y(i) \leq y] = \frac{\prod_{\substack{j=1 \\ (j \neq i)}}^{M} (1 - e^{-g_j y} + e^{-g_j a}) - \gamma_i}{1 - \gamma_i}; \qquad 0 \leq y \leq a. \tag{33}$$

Now, (8) can be written as

$$X = \sum_{n=1}^{\infty} \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_n=1}^{M} \left[\prod_{k=1}^{n-1} \nu_{i_k}(1-\gamma_{i_k})\right] \nu_{i_n}\gamma_{i_n} \cdot \left\{\sum_{k=1}^{n-1} [I^{(k)} + 1 + a + Y(i_k)] + I^{(n)} + 1 + a\right\} \tag{34}$$

where $\nu_{i_k}(1 - \gamma_{i_k})$ is the probability that the kth transmission period is initiated by user i_k and that it involves collision. Equation (34) for X can be simplified by using (30) and (32) as follows:

$$X = \sum_{n=1}^{\infty} \left\{(1-\gamma)^{n-2}\gamma \sum_{k=1}^{n-1} \sum_{i_k=1}^{M} [I^{(k)} + 1 + a + Y(i_k)]\nu_{i_k}(1-\gamma_{i_k}) + (1-\gamma)^{n-1}\gamma[I^{(n)} + 1 + a]\right\}. \tag{35}$$

Let us define a random variable Y by

$$\text{Prob } [Y \leq y] = \frac{1}{1-\gamma} \sum_{i=1}^{M} \nu_i(1-\gamma_i) \cdot \text{Prob } [Y(i) \leq y] \qquad 0 \leq y \leq a. \tag{36}$$

Then we have

$$X = \sum_{n=1}^{\infty} (1-\gamma)^{n-1}\gamma \cdot \left\{\sum_{k=1}^{n-1} [I^{(k)} + 1 + a + Y] + I^{(n)} + 1 + a\right\}. \tag{37}$$

Note that (37) is of the same form as (8) conditionally summed. Therefore, (12) for $X^*(s)$ still holds when T and F are given by

$$T = 1 + a; \quad F = 1 + a + Y \tag{38}$$

and Y in (36) is used. Thus we get

$$\bar{X}=\frac{1}{\gamma}(\bar{I}+1+a)+\left(\frac{1}{\gamma}-1\right)\bar{Y}$$

$$\text{Var}[X]=\frac{1}{\gamma}\text{Var}[I]+\left(\frac{1}{\gamma}-1\right)\text{Var}[Y]+(\bar{I}+1+a+\bar{Y})^2\frac{1-\gamma}{\gamma^2} \quad (39)$$

where

$$\bar{Y}=\sum_{i=1}^{M}\overline{Y(i)}\nu_i(1-\gamma_i)/(1-\gamma)$$

$$=\frac{1}{1-\gamma}\int_0^a\left[1-\sum_{i=1}^{M}\nu_i\prod_{\substack{j=1\\(j\neq i)}}^{M}(1-e^{-g_jy}+e^{-g_ja})\right]dy$$

$$\text{Var}[Y]=\sum_{i=1}^{M}\overline{Y(i)^2}\nu_i(1-\gamma_i)/(1-\gamma)-\overline{Y}^2. \quad (40)$$

Especially, we have explicitly

$$S=\frac{\gamma}{\dfrac{1}{\sum_{i=1}^{M}g_i}+1+2a-\sum_{i=1}^{M}\nu_i\int_0^a\left[\prod_{\substack{j=1\\(j\neq i)}}^{M}(1-e^{-g_jy}+e^{-g_ja})\right]dy}. \quad (41)$$

Note that q_i, the probability that a successful transmission is achieved by user i, is now given by

$$q_i=\frac{g_i\gamma_i}{\sum_{j=1}^{M}g_j\gamma_j}=\frac{\nu_i\gamma_i}{\gamma} \qquad i=1,2,\cdots,M. \quad (42)$$

Thus, the throughput of user i is given by

$$S_i=q_iS=\frac{\nu_i\gamma_i}{\dfrac{1}{\sum_{j=1}^{M}g_j}+1+2a-\sum_{j=1}^{M}\nu_j\int_0^a\left[\prod_{\substack{k=1\\(k\neq j)}}^{M}(1-e^{-g_ky}+e^{-g_ka})\right]dy} \qquad i=1,2,\cdots,M. \quad (43)$$

In the case of identical users ($g_i = g$ for all i), we have the channel throughput of unslotted CSMA as

$$S=\frac{e^{-ga(M-1)}}{\dfrac{1}{gM}+1+2a-\int_0^a(1-e^{-gy}+e^{-ga})^{M-1}\,dy}. \quad (44)$$

For $M\to\infty$ with G fixed at $G = gM$, since $(1-e^{-gy}+e^{-ga})^{M-1}\cong[1-g(a-y)]^{M-1}\cong e^{-G(a-y)}$, (44) reduces to

$$S=\frac{Ge^{-aG}}{G(1+2a)+e^{-aG}} \quad (45)$$

given in [4] and [10]. In this limit, the distribution and variance of X are given by

$$X^*(s)=\frac{G(s-G)e^{-s(1+a)-aG}}{(s+G)(s-G)-G^2e^{-aG-s(1+a)}[1-e^{-(s-G)a}]}$$

$$\text{Var}[X]=\frac{2-e^{-aG}}{G^2e^{-aG}}+\frac{(1+2a)^2}{e^{-2aG}}-\frac{1+2a}{e^{-aG}}. \quad (46)$$

V. Unslotted CSMA with Collision Detection

The case of unslotted CSMA with collision detection can be treated similarly except for the duration of an unsuccessful transmission period, which is now expressed as

$$F=b+a+Y_1 \quad (47)$$

where b is the time required for an idle-period-breaking user to abort its transmission after the first colliding packet has started transmission, and Y_1 is the time offset of the first colliding transmission; see Fig. 2 for the channel timing chart (adapted from [11] for the unslotted system) and see Fig. 1(d) for the packet interdeparture process. We assume that $a \leqslant b \lesssim 1$; the case $b = 1$ in an unslotted system is not equivalent to the one without collision detection. (Our proposition for the duration of F given in (47) differs from that of Molle [6], who used $F = b + a + Y$ where Y is the transmission start time of the last colliding packet. As shown in Fig. 2, it is the first colliding packet that stops the transmission of the leading packet lingering until the last; other transmissions have been aborted before by detecting the leading packet. Thus, (47) seems more reasonable although the difference is of order a.)

The analysis in Section IV can be readily applied to unslotted CSMA with collision detection. Instead of (37), we have

$$X=\sum_{n=1}^{\infty}(1-\gamma)^{n-1}\gamma\left\{\sum_{k=1}^{n-1}[I^{(k)}+b+a+Y_1]+I^{(n)}+1+a\right\} \quad (48)$$

where Y_1 is a random variable defined by

$$\text{Prob}[Y_1\leqslant y]=\frac{1}{1-\gamma}\sum_{i=1}^{M}\nu_i(1-\gamma_i)\cdot\text{Prob}[Y_1(i)\leqslant y] \qquad 0\leqslant y\leqslant a \quad (49)$$

and $Y_1(i)$ is the transmission start time of the first colliding packet in an unsuccessful transmission period initiated by user i. In (48) and (49), $I^{(k)}$ is the kth channel idle period duration, and ν_i, γ, and γ_i are given in (30)–(32). The distribution of $Y_1(i)$ is given by

$$\text{Prob}[Y_1(i)>y]=\frac{\exp\left[-y\sum_{\substack{j=1\\(j\neq i)}}^{M}g_j\right]-\gamma_i}{1-\gamma_i} \qquad 0\leqslant y\leqslant a \quad (50)$$

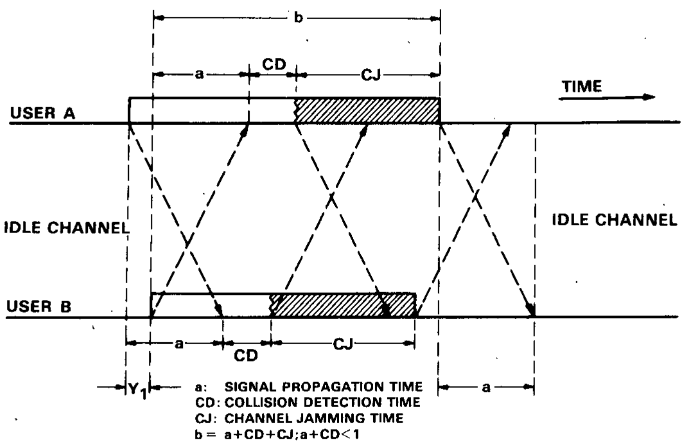

Fig. 2. Collision detection timing in unslotted CSMA/CD.

from which we have

$$\bar{Y}_1=\frac{1}{1-\gamma}\sum_{i=1}^{M}\nu_i(1-\gamma_i)\Big(\sum_{\substack{j=1\\(j\neq i)}}^{M}g_j\Big)^{-1}-\frac{a\gamma}{1-\gamma}$$

$$\text{Var}\ [Y_1]=\frac{1}{1-\gamma}\sum_{i=1}^{M}\nu_i(1-\gamma_i)\,\overline{Y_1(i)^2}-\bar{Y}_1^2 \quad (51)$$

where

$$\overline{Y_1(i)^2}=\frac{2}{\Big(\sum_{\substack{j=1\\(j\neq i)}}^{M}g_j\Big)^2}-\frac{a^2\gamma_i}{1-\gamma_i}-\frac{2a\gamma_i}{(1-\gamma_i)\Big(\sum_{\substack{j=1\\(j\neq i)}}^{M}g_j\Big)}.$$

Therefore, we can compute

$$\bar{X}=\Big(\frac{1}{\gamma}-1\Big)(\bar{I}+b+a+\bar{Y}_1)+\bar{I}+1+a$$

$$\text{Var}\ [X]=\frac{1}{\gamma}\ \text{Var}\ [I]+\Big(\frac{1}{\gamma}-1\Big)\ \text{Var}\ [Y_1]+(\bar{I}+b+a+\bar{Y}_1)^2\,\frac{1-\gamma}{\gamma^2}. \quad (52)$$

We have the channel throughput

$$S=\frac{\gamma}{\dfrac{1}{\sum_{i=1}^{M}g_i}+\gamma+(1-\gamma)(b+a)+\sum_{i=1}^{M}\dfrac{\nu_i(1-\gamma_i)}{\sum_{\substack{j=1\\(j\neq i)}}^{M}g_j}} \quad (53)$$

and the throughput for user i

$$S_i=q_iS=\frac{\nu_i\gamma_i}{\dfrac{1}{\sum_{j=1}^{M}g_j}+\gamma+(1-\gamma)(b+a)+\sum_{j=1}^{M}\dfrac{\nu_j(1-\gamma_j)}{\sum_{\substack{k=1\\(k\neq j)}}^{M}g_k}}$$

$$i=1,\ 2,\ \cdots,\ M. \quad (54)$$

In the case of identical users ($g_i = g$ for all i), we have

$$X^*(s)=\frac{e^{-ga(M-1)-s(1+a)}}{\dfrac{s+gM}{gM}-g(M-1)e^{-s(b+a)}\cdot\dfrac{1-e^{-[s+g(M-1)]a}}{s+g(M-1)}}$$

$$S=\frac{e^{-ga(M-1)}}{\dfrac{1}{gM}+e^{-ga(M-1)}+\Big[b+a+\dfrac{1}{g(M-1)}\Big][1-e^{-ga(M-1)}]}$$

$$\text{Var}\ [X]=\frac{1}{\gamma(gM)^2}+\Big(\frac{1}{\gamma}-1\Big)\ \text{Var}\ [Y_1]+\Big(\frac{1}{gM}+b+a+\bar{Y}_1\Big)^2\cdot\frac{1-\gamma}{\gamma^2} \quad (55)$$

where

$$\bar{Y}_1=\frac{1}{g(M-1)}-\frac{a\gamma}{1-\gamma};$$

$$\text{Var}\ [Y_1]=\frac{1}{g^2(M-1)^2}-\frac{a^2\gamma}{(1-\gamma)^2};\quad \gamma=e^{-ga(M-1)}. \quad (56)$$

The value of g which maximizes S in (55) is given as a solution to the equation

$$(2M-1)[1-ga(M-1)]=\Big(1+\frac{b}{a}\Big)(ga)^2M(M-1)^2+Me^{-ga(M-1)}. \quad (57)$$

For $M \to \infty$ with G fixed at $G = gM$, we have

$$X^*(s)=\frac{G(s+G)e^{-aG-s(1+a)}}{(s+G)^2-G^2e^{-s(b+a)}[1-e^{-(s+G)a}]}$$

$$S=\frac{Ge^{-aG}}{2+(G-1)e^{-aG}+(b+a)G(1-e^{-aG})}. \quad (58)$$

The optimal G for (58) is similarly determined by

$$2(1-aG)=\Big(1+\frac{b}{a}\Big)(aG)^2+e^{-aG}. \quad (59)$$

VI. Remarks and Numerical Results

In unslotted CSMA and unslotted CSMA/CD, we have derived $X^*(s)$, S, and Var $[X]$ for systems of identical users as special cases of generally nonidentical users. However, it is possible to consider the cases of identical users using the formulation of Section III. This treatment is shown in [8]. We have also omitted the presentation of trivial analysis for slotted ALOHA.

In Fig. 3(a), we plot the throughput values for the protocols we have studied when they are maximized by optimizing the transmission parameters (e.g., p and g). The results shown are for the limit $M \to \infty$ while holding pM or gM at a fixed finite value. (The curves for CSMA/CD are for the ideal case: $b = a$.) For proper comparison between ALOHA and CSMA in an environment of nonzero propagation delay ($a > 0$), we have uniformly assumed that the duration of a successful transmission period is $1 + a$. Thus, the plots for ALOHA systems are $S/(1 + a)$ where $S = 1/e$ for slotted ALOHA and $S = 1/(2e)$ for pure ALOHA. The throughput for perfect scheduling is similarly assumed to be $1/(1 + a)$. These maximal throughput

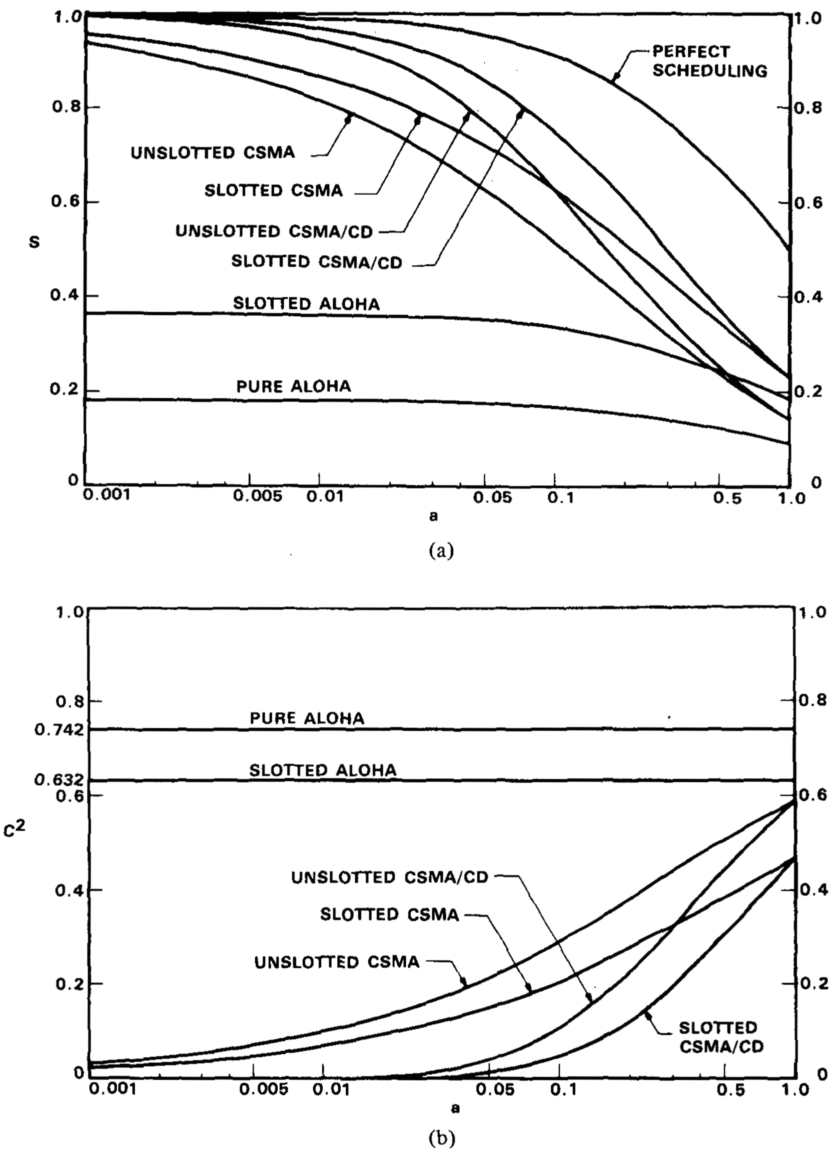

Fig. 3. (a) The (maximized) channel throughput S. (b) The coefficient of variation C^2 for packet interdeparture time.

curves have been studied in [4], [6], [10]. The new results in the present paper are the corresponding plots in Fig. 3(b) of the coefficient of variation of the packet interdeparture times. It is remarkable that (at maximal throughputs) they are all below 1. Specifically, in efficient CSMA cases they are almost 0, which implies that the channel service time is nearly constant.

The throughput S and C^2 for slotted CSMA/CD are displayed in Fig. 4 with several values of collision detection time b ($a \leqslant b \leqslant 1$). Again, for a typical example of a local-area computer network [11] where a is from 0.01 to 0.1 and b is short (< 0.1, say), we have very small values of C^2, despite the fact that throughput is more or less degraded. This observation suggests an $M/D/1$ queueing model (with service rate equated to the given channel throughput) approximation to the queue length distribution.

VII. Conclusion

We have studied the packet departure processes in a variety of contention-type packet broadcasting systems with the heavy traffic assumption. The channel access protocols considered include both slotted and unslotted systems of ALOHA and carrier-sense-multiple-access (CSMA). The effects of collision detection on CSMA have also been investigated.

Through the analyis of channel activity cycles alternating between the idle and (successfully or unsuccessfully) transmitting states, we have derived the distribution of the packet interdeparture time X. Then we found the channel throughput ($S = 1/\bar{X}$) and the coefficient of variation of X ($C^2 =$ Var $[X]/\bar{X}^2$) explicitly. All the results for the distribution of X and C^2 are new. Some results for S [specifically (21), (41), (44), (53), (55), and (58)] are also newly derived in this paper. It has been shown that in efficient CSMA systems with collision detection, C^2 is very small, while the throughput suffers some degradation.

Using $\bar{X}$ and C^2 together with the elementary renewal theorem, we have also obtained the asymptotic behavior of the number of successful transmissions at interfering individual queues. These results can be used to determine the coefficients in the diffusion process approximation to the queue length distribution at users [9].

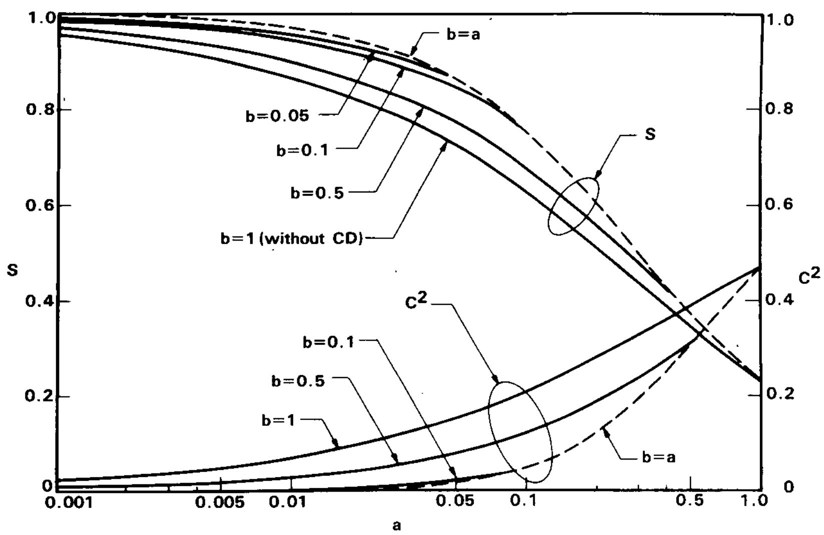

Fig. 4. The channel throughput (S) and coefficient of variation (C^2) for packet interdeparture time in slotted CSMA/CD.

APPENDIX A

DERIVATION OF (4)

Let $F(z)$ be the z-transform of the distribution of $D(t)$, the number of departures from all users during $[0, t]$. The joint probability that during this interval user i attains $D_i(t) = k_i$ departures ($i = 1, 2, \cdots, M$), given that $D(t) = k$, is given by a multinominal distribution

$$\frac{k!}{k_1! k_2! \cdots k_M!} q_1{}^{k_1} q_2{}^{k_2} \cdots q_M{}^{k_M} \tag{A.1}$$

where $k = k_1 + k_2 + \cdots + k_M$, and q_i is the probability that a departure is achieved by user i. Thus, the conditional joint z-transform of $\boldsymbol{D}(t) = [D_1(t), D_2(t), \cdots, D_M(t)]$ is given by

$$(q_1 z_1 + q_2 z_2 + \cdots + q_M z_M)^k \tag{A.2}$$

and the unconditional joint z-transform of $\boldsymbol{D}(t)$ is given by a compound distribution

$$G(\boldsymbol{z}) = F(q_1 z_1 + q_2 z_2 + \cdots + q_M z_M) \tag{A.3}$$

where $\boldsymbol{z} = [z_1, z_2, \cdots, z_M]$.

Now, from (A.3) and the definition of $F(z)$, the means and covariances of $\boldsymbol{D}(t)$ can be formally calculated. First, the mean is given by

$$\overline{D_i(t)} = \left.\frac{\partial G(\boldsymbol{z})}{\partial z_i}\right|_{\boldsymbol{z}=\boldsymbol{1}} = q_i \left.\frac{dF(z)}{dz}\right|_{z=1} = q_i \overline{D(t)} \qquad i = 1, 2, \cdots, M \tag{A.4}$$

where $\boldsymbol{1} = [1, 1, \cdots, 1]$. Next, from

$$\overline{D_i(t)^2} - \overline{D_i(t)} = \left.\frac{\partial^2 G(\boldsymbol{z})}{\partial z_i^2}\right|_{\boldsymbol{z}=\boldsymbol{1}} = q_i^2 [\overline{D(t)^2} - \overline{D(t)}] \tag{A.5}$$

the variance is given by

$$\begin{aligned} \text{Var } [D_i(t)] &\triangleq \overline{D_i(t)^2} - [\overline{D_i(t)}]^2 \\ &= q_i^2 [\overline{D(t)^2} - \overline{D(t)}] + q_i \overline{D(t)} - q_i^2 [\overline{D(t)}]^2 \\ &= q_i^2 \text{ Var } [D(t)] + q_i (1 - q_i) \overline{D(t)} \\ &\qquad i = 1, 2, \cdots, M. \end{aligned} \tag{A.6}$$

Also, for $i \neq j$, from

$$\overline{D_i(t) \cdot D_j(t)} = \left.\frac{\partial^2 G(\boldsymbol{z})}{\partial z_i \partial z_j}\right|_{\boldsymbol{z}=\boldsymbol{1}} = q_i q_j [\overline{D(t)^2} - \overline{D(t)}] \tag{A.7}$$

the covariance is given by

$$\begin{aligned} \text{Cov } [D_i(t), D_j(t)] &\triangleq \overline{D_i(t) \cdot D_j(t)} - \overline{D_i(t)} \cdot \overline{D_j(t)} \\ &= q_i q_j [\overline{D(t)^2} - \overline{D(t)}] - q_i q_j [\overline{D(t)}]^2 \\ &= q_i q_j \{\text{Var } [D(t)] - \overline{D(t)}\} \\ &\qquad i \neq j, \; i, j = 1, 2, \cdots, M. \end{aligned} \tag{A.8}$$

Equations (A.4), (A.7), and (A.8) give (4).

APPENDIX B

DERIVATION OF (15)

In an infinite population of pure ALOHA users, the duration of an unsuccessful transmission period F consists of an indefinite number (L, say) of packet interarrival times whose durations are less than 1 (denoted by $f^{(1)}, f^{(2)}, \cdots, f^{(L)}$) terminated by a full length of 1 (see Fig. 5):

$$F = f^{(1)} + f^{(2)} + \cdots + f^{(L)} + 1. \tag{B.1}$$

All $f^{(n)}$'s are independent and identically distributed [let their generic representation be f with its pdf's Laplace transform $f^*(s)$] as

$$\begin{aligned} &\text{Prob } [f \leqslant t] = \frac{1 - e^{-Gt}}{1 - e^{-G}} \qquad 0 \leqslant t \leqslant 1 \\ &f^*(s) = \frac{G[1 - e^{-(s+G)}]}{(s+G)(1 - e^{-G})} \\ &\bar{f} = \frac{1}{G} - \frac{e^{-G}}{1 - e^{-G}}; \quad \text{Var } [f] = \frac{1}{G^2} - \frac{e^{-G}}{(1 - e^{-G})^2} \end{aligned} \tag{B.2}$$

where G is the rate of arrivals. The number of such interarrival times L [with its distribution's z-transform $L^*(z)$]

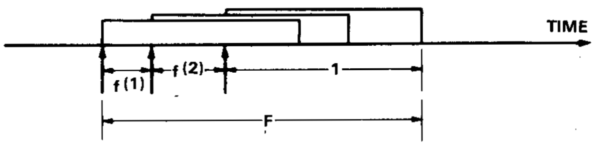

Fig. 5. Unsuccessful transmission period in pure ALOHA.

is independent of $f^{(n)}$'s and is geometrically distributed as

$$\text{Prob}\ [L=n]=(1-e^{-G})^{n-1}e^{-G} \qquad n=1,\ 2,\ \cdots$$

$$L^*(z)=\frac{ze^{-G}}{1-z(1-e^{-G})}$$

$$\bar{L}=e^G;\quad \text{Var}\ [L]=e^G(e^G-1). \tag{B.3}$$

From (B.1)–(B.3), we have

$$F^*(s)=e^{-s}L^*[f^*(s)]=\frac{Ge^{-(s+G)}[1-e^{-(s+G)}]}{(1-e^{-G})[s+Ge^{-(s+G)}]}$$

$$\bar{F}=\bar{L}\bar{f}+1=\frac{e^G-1-Ge^{-G}}{G(1-e^{-G})}$$

$$\text{Var}\ [F]=\bar{L}\ \text{Var}\ [f]+(\bar{f})^2\ \text{Var}\ [L]$$

$$=\frac{e^{2G}}{G^2}-\frac{2e^G}{G}-\frac{e^{-G}}{(1-e^{-G})^2} \tag{B.4}$$

which is identical to (15).

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